# Probability theory
# Lesson 20
# The Binomial Distribution

## 20.1 - What is a Binomial Distribution?

**20.1 - Problem 1:**
Since five bottles are checked: n = 5.

We assume each bottle checked are independent.

Success on each trial is defined as a bottle contains more then 16 ounces: p = 0.10.

Failure on each trial is defined as: a bottle does not contain more than 16 ounces:

q = 1- 0.10 = 0.90.

The number of possible bottles with more then 16 ounces ranges from zero to 5:
k = 0,1,2,3,4,5.

The distribution of this binomial distribution is

$$P\{X = 0\} = \binom{5}{0}(0.10)^0(0.90)^5 \approx 0.5905$$

$$P\{X = 1\} = \binom{5}{1}(0.10)^1(0.90)^4 \approx 0.3281$$

$$P\{X = 2\} = \binom{5}{2}(0.10)^2(0.9)^3 \approx 0.0729$$

$$P\{X = 3\} = \binom{5}{3}(0.10)^3(0.90)^2 \approx 0.0081$$

$$P\{X = 4\} = \binom{5}{4}(0.10)^4(0.90)^1 \approx 0.0005$$

$$P\{X = 5\} = \binom{5}{5}(0.10)^5(0.90)^0 \approx 0.00001$$

**20.1 - Problem 2:**

➤(a).

Since 15 men are tested: n = 15.

We assume the tests between the men are independent.

Success on each trial is defined as detecting positive coronary blockage: p = 0.80.

Failure on each trial is defined as not detecting positive coronary blockage: q = 1- 0.80 = 0.20.

To find the probability that 12 out of the 15 men tested positive we use the formula

$$P\{X = 12\} = \binom{15}{12}(0.80)^{12}(0.20)^3 \approx 0.2501$$

➤(b).

To find the probability that 10 out of the 15 men tested negative is the same as the probability that 5 of the 15 men tested positive. We use the formula

$$P\{X = 5\} = \binom{15}{5}(0.80)^5(0.20)^{10} \approx 0.0001$$

➤(c).

Here we assume p = 0.20. and we will evaluate the binomial formula for k = 7,8,9,10.

$$P\{7 \le X \le 10\} = P\{X = 7\} + P\{X = 8\} + P\{X = 9\} + P\{X = 10\}$$

$$\binom{15}{7}(0.20)^7(0.80)^8 + \binom{15}{8}(0.20)^8(0.80)^7 + \binom{15}{9}(0.20)^9(0.80)^6 +$$

$$\binom{15}{10}(0.20)^{10}(0.80)^5 \approx 0.0138 + 0.0035 + 0.0007 + 0.0001 = 0.0181$$

➤(d).

The event that at least one man tested positive equals $\{1 \le X\} = \{X = 0\}'$.

$$P\{X = 0\} = \binom{15}{0}(0.8)^0(0.20)^{15} \approx 0$$

Therefore,

$$P\{1 \le X\} = P\{X = 0\}' \approx 1 - P\{X = 0\} = 1 - 0 = 1.$$

➤(e).
Here we use p = 0.20 and write the event that at most 13 men tested negative as

$\{X \le 13\} = \{X \ge 14\}'$.

$P\{X \ge 14\} = P\{X = 14\} + P\{X = 15\} = \binom{15}{14}(0.20)^{14}(0.80)^{1} + \binom{15}{15}(0.20)^{15}(0.80)^{0} \approx$

$0 + 0 = 0$

Therefore,

$P\{X \le 13\} = P\{X \ge 14\}' = 1 - P\{X \ge 14\}' \approx 1 - 0 = 1.$

**20.1 - Problem 3:**
Step 1: First we must find the probability that the student will answer at least 7 of the questions correctly.

This is a binomial distribution where X is the number of questions answered correctly, N = 10, and

p = 0.25,

the probability that a question is answered correctly (1 out of 4).

Step 2: E: the event that at least 7 of the questions are answered correctly.

$P(E) = P\{X \ge 7\} = P\{X = 7\} + P\{X = 8\} + P\{X = 9\} + P\{X = 10\} =$

$$\binom{10}{7}(0.25)^{7}(0.75)^{3} + \binom{10}{8}(0.25)^{8}(0.75)^{2} + \binom{10}{9}(0.25)^{9}(0.75)^{1} + \binom{10}{10}(0.25)^{10}(0.75)^{0} \approx$$

0.0035

Step 3: To find the probability that the student passes none of the exams, let X be the number of exams passee. Since it is reasonable to assume that passes any of these exams are independent of each other, we assume a binomial distribution where

N = 8, p = 0.0035. k = 0.

$P\{X = 0\} = \binom{8}{0}(0.0035)^{0}(0.9965)^{8} \approx 0.97,$

the probability that the student will pass none of the tests.

# 20.2- The Mean and Variance of the Binomial Distribution

**20.2 - Problem 1**:
Here we define success as a word is misspelled. It is also reasonable to assume an independence in typing errors among the words. Therefore, we accept this as a binomial sample space where

$N = 120{,}000$, and $p = 0.05$.

$\mu = Np = (120000)(0.05) = 6{,}000.$

$\sigma^2 = Np(1 - p) = (120000)(0.05)(0.95) = 5{,}700.$

## Supplementary Problems

**1.**
We assume that each of the 6 hands are independent of each other.

Step 1: We need to find p the probability that a hand contains exactly two spades.

$$p = \frac{\binom{13}{2}\binom{39}{3}}{\binom{52}{5}} = \frac{(78)(9{,}139)}{2{,}598{,}960} = \frac{712{,}842}{2{,}598{,}960} \approx 0.274$$

Step 2: For each of the 6 hands, we define success as a hand contains exactly two spades. Since we have a binomial distribution where

$N = 6$, $p \approx 0.274$ and $k = 3$.

$$P\{X = 3\} = \binom{6}{3}(0.274)^3\ (0.726)^3 \approx 0.16$$

**2.**
From problem 1, we have $p \approx 0.274$.

Step 1: We let $k = 2$, the number of hands each consisting of exactly two spades.

Step 2: Let N equal the number of hands selected so that the chance is greater then 0.80 that he gets at least two hands each containing of exactly two spades.

Step 3: We need to solve the inequality

$P\{X \geq 2\} = 1 - P\{X \leq 1\} = 1 - P\{X = 0\} - P\{X = 1\} > 0.80$

for different values of N and where X is the number of hands that contain exactly two spades.

N = 5:

$$P\{X = 0\} = \binom{5}{0}(0.274)^0(0.726)^5 \approx 0.20$$

$$P\{X = 1\} = \binom{5}{1}(0.274)^1(0.726)^4 \approx 0.38$$

$P\{X \geq 2\} = 1 - 0.20 - 0.38 = 0.42$

N = 7:

$$P\{X = 0\} = \binom{7}{0}(0.274)^0(0.726)^7 \approx 0.11$$

$$P\{X = 1\} = \binom{7}{1}(0.274)^1(0.726)^6 \approx 0.28$$

$P\{X \geq 2\} = 1 - 0.11 - 0.28 = 0.61$

N = 9:

$$P\{X = 0\} = \binom{9}{0}(0.274)^0(0.726)^9 \approx 0.06$$

$$P\{X = 1\} = \binom{9}{1}(0.274)^1(0.726)^8 \approx 0.19$$

$P\{X \geq 2\} = 1 - 0.06 - 0.19 = 0.75$

N = 10:

$$P\{X = 0\} = \binom{10}{0}(0.274)^0(0.726)^{10} \approx 0.04$$

$$P\{X = 1\} \approx \binom{10}{1}(0.274)^1(0.726)^9 \approx 0.15$$

$$P\{X \geq 2\} = 1 - 0.04 - 0.15 = 0.81$$

Therefore, if he draws 10 hands, there is a chance better then 80% that at least two of these hands will have exactly two spades.

**3.**

Let $N \geq 4$ equal the number of hands drawn.

We evaluate the formula for various values of N

$$P\{X = 4\} = \binom{N}{4}(0.274)^4(0.726)^{N-4}$$

where X = 4 is the number of hands that contain exactly 2 spades

N = 4

$$P\{X = 4\} = \binom{4}{4}(0.274)^4(0.726)^0 \approx 0.0056$$

N = 6

$$P\{X = 6\} = \binom{6}{4}(0.274)^4(0.726)^2 \approx 0.04$$

N = 8:

$$P\{X = 8\} = \binom{8}{4}(0.274)^4(0.726)^4 \approx 0.11$$

N = 10:

$$P\{X = 10\} = \binom{10}{4}(0.274)^4(0.726)^6 \approx 0.17$$

N = 12:

$$P\{X = 12\} = \binom{12}{4}(0.274)^4(0.726)^8 \approx 0.21$$

N = 14:

$P\{X = 14\} = \binom{14}{4}(0.274)^4(0.726)^{10} \approx 0.228$

N = 15:

$P\{X = 15\} = \binom{15}{4}(0.274)^4(0.726)^{11} \approx 0.226$

N = 16:

$P\{X = 16\} = \binom{16}{4}(0.274)^4(0.726)^{12} \approx 0.219$

Therefore, N = 16 hands gives the best chance, 0.219, of getting 4 hands with exactly 2 spaces.

**4.**
Since it is reasonable to assume a binomial distribution, we must first determine the probability p that a randomly selected transmission is defective.

Step 1:
**D**: the event a randomly selected transmission is defective.
**E**: the event a randomly selected transmission came from the East coast.

$\mathbf{D} = (\mathbf{D}\cap\mathbf{E})\cup(\mathbf{D}\cap\mathbf{E}')$

$P(\mathbf{D}) = P(\mathbf{D}\cap\mathbf{E}) + P(\mathbf{D}\cap\mathbf{E}') = P(\mathbf{D}|\mathbf{E})P(\mathbf{E}) + P(\mathbf{D}|\mathbf{E}')P(\mathbf{E}') = (0.05)(0.70) + (0.07)(0.30) =$

$0.056 = p$

Step 2:

**F**: the event that at least one of the five selected is defective.
**F′**: the event that none of the five selected are defective.

$P(\mathbf{F}') = \binom{5}{0}(0.056)^0(0.944)^5 \approx 0.75$

$\mathbf{P(F)} = 1 - P(F') = 1 - 0.75 = 0.25.$

**5.**
$X_1$: the random variable that equals the number of times the number 1 appears.

$X_2$: the random variable that equals the number of times the number 2 appears.

$X_3$: the random variable that equals the number of times the number 3 appears.

$X_4$: the random variable that equals the number of times the number 4 appears.

$X_5$: the random variable that equals the number of times the number 5 appears.

$X_6$: the random variable that equals the number of times the number 6 appears.

$P\{X_1 = 2; X_2 = 2; X_3 = 2; X_4 = 2; X_5 = 2; X_6 = 2\} =$

$$\binom{12}{2}\binom{10}{2}\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2}\frac{1}{6^2}\frac{1}{6^2}\frac{1}{6^2}\frac{1}{6^2}\frac{1}{6^2}\frac{1}{6^2} = \frac{7484400}{2176782336} = 0.0034383$$

**6.**

➤a.

We use the multinomial distribution of these binomial variables is

$$P\{X_1 = k_1; X_2 = k_2;...; X_r = k_r\} = \binom{N}{k_1}\binom{N_1}{k_2} \cdots \binom{N_{r-1}}{k_r} p_1^{k_1} P_2^{k_2} p_3^{k_3} ... p_r^{k_r}$$

where $N_1 = N - k_1$, $N_2 = N - k_1 - k_2$,... $N_{r-1} = N - k_1 - k_2 - ... - k_r$ and $k_1 + k_2 + ... + k_r = N$.

$X_1$: the random variable that equals the number of Democrats.

$X_2$: the random variable that equals the number of Republicans.

$X_3$: the random variable that equals the number of Independents.

$X_4$: the random variable that equals the number of Libertarians.

$X_5$: the random variable that equals the number of Greens.

$P\{X_1 = 3; X_2 = 4; X_3 = 0; X_4 = 2; X_4 = 1\} =$

$$\binom{10}{3}\binom{7}{4}\binom{3}{0}\binom{3}{2}\binom{1}{1}(0.35)^3(0.40)^4(0.15)^0(0.07)^2(0.03)^1 \approx 0.00203$$

➤b.

The model we use us binomial. Here X is the number of Republicans where

k = 6,
N = 10,
p = 0.40,
q = 0.60.

$$P\{X = 6\} = \binom{10}{6}(0.4)^6(0.6)^4 \approx 0.11$$

➤c.
The model we use us binomial. Here X is the number of Democrats where

k = 3,
N = 10,
p = 0.35,
q = 0.65.

$$P\{X = 3\} = \binom{10}{3}(0.35)^3(0.65)^7 \approx 0.25$$

➤d.
Step 1:
**R**: the event that six are Republicans.
**D**: the event that three are Democrats.
**E**: the event that Six are Republicans or three are Democrats.

**E** = **R**∪**D**

Step 2: P(**E**) = P(**R**∪**D**) = **P(R)** + P(**D**) - **P(R**∩**D**)

From (c) and (d) we have

P(**E**) = P(**R**∪**D**) = **P(R)** + P(**D**) - **P(R**∩**D**) = 0.11 + 0.25 - P(R∩D)

Step 3: Here, we use the multinomial distribution of these binomial variables is

$$\mathbf{P(R \cap D)} = P\{X_1 = 6; X_2 = 3; X_3 = 1\} = \binom{10}{6}\binom{4}{3}\binom{1}{1}(0.40)^6(0.35)^3(0.25)^1 \approx 0.0367$$

where

$X_1$: the random variable that equals the number of Republicans.
$X_2$: the random variable that equals the number of Democrats.

$X_3$: the random variable that equals the number of other party members.

Step 4: P(**E**) = 0.11 + 0.25 - P(**R**∩**D**) = 0.11 + 0.25 - 0.0367 ≈ 0.32

**7.**
Step 1:
{X = 5}: the event that the red die results in 5 even numbers.
{Y = 3}: the event that the blue die results in 3 odd numbers.
**E**: the event that the red die results in 5 even numbers or the blue die results in 3 odd numbers.

**E** = {X = 5}∪{Y = 3}

Step 2: P(**E**) = P{X = 5} + P{Y = 3} - P[{X = 5}∩{Y = 3}]

Because of independence,

P[{X = 5}∩{Y = 3}] = P{X = 5}P{Y = 3} and

P(**E**) = P{X = 5} + P{Y = 3} - P{X = 5}P{Y = 3}

Step 3:

$$P\{X = 5\} = \binom{10}{5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^5 \approx 0.08$$

$$P\{Y = 3\} = \binom{10}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^7 \approx 0.12$$

P(E) = P{X = 5} + P{Y = 3} - P{X = 5}P{Y = 3} ≈ 0.08 + 0.12 - (0.08)(0.12) ≈ 0.19

**8.**
➤a.
The sample space is

S = {(3),(f,3),(f,f,3),(f,f,f,3),...},

where f represents any number on the die other then 3. This is not a binomial sample space since order is required.

➤b.
The random variable X represents the number of tosses to complete the game.

For example,

X(f,f,f,f,f,3) = 6 and

$$P\{X = 6\} = (\frac{5}{6})^5(\frac{1}{6}).$$

For an integer k > 0, we have

$$P\{X = k\} = (\frac{5}{6})^{k-1}(\frac{1}{6}).$$

Therefore,

$$P\{X > 4\} = 1 - P\{X \le 4\} = 1 - [P\{X = 0\} + P\{X = 1\} + P\{X = 2\} + P\{X = 3\} + P\{X = 4\}]$$

$$= 1 - [(\frac{5}{6})^0(\frac{1}{6}) + (\frac{5}{6})^1(\frac{1}{6}) + (\frac{5}{6})^2(\frac{1}{6}) + (\frac{5}{6})^3(\frac{1}{6}) + (\frac{5}{6})^4(\frac{1}{6})] =$$

$$1 - \frac{1}{6}[(\frac{5}{6})^0 + (\frac{5}{6})^1 + (\frac{5}{6})^2 + (\frac{5}{6})^3 + (\frac{5}{6})^4] = \frac{3125}{7776}$$

**9.**

➤a.

Let a = 1/6 , the probability that a 3 occurs.

Let b = 5/6, the probability that a 3 does not occurs.

Let $X_k$ equal the event that a 3 occurs on the kth toss.

**E**: the event that an even number of 3s' occurs.

**E** = $\{X = 0\} \cup \{X = 1\} \cup \{X = 2\} \cup ... \cup \{X = 10\}$

P( **E**) = $P[\{X = 0\} \cup \{X = 1\} \cup \{X = 2\} \cup ... \cup \{X = 10\}] =$

$P\{X = 0\} + P\{X = 1\} + P\{X = 2\} + ... + P\{X = 10\} =$

$$\binom{10}{0}(\frac{1}{6})^0(\frac{5}{6})^{10} + \binom{10}{2}(\frac{1}{6})^2(\frac{5}{6})^8 + \binom{10}{4}(\frac{1}{6})^4(\frac{5}{6})^6 + ... + \binom{10}{10}(\frac{1}{6^{10}})(\frac{5}{6^0}) =$$

$$\frac{1}{2}\{(\frac{1}{6} + \frac{5}{6})^{10} + (\frac{1}{6} - \frac{5}{6})^{10}\} = \frac{60073}{118098}$$

➤b.
E′: the event that an odd number of 3s' occur.

$$P(\mathbf{E}') = 1 - P(\mathbf{E}) = 1 - \frac{60073}{118098} = \frac{58025}{118098}$$

**10.**
Let X equal the number of chips found defective. We need to find p so that $P\{X \geq 2\} = 0.01$.

We know $P\{X \geq 2\} = 1 - P\{X \leq 1\}$.

$P\{X \leq 1\} = 1 - P\{X \geq 2\} = 0.99$.

$P\{X \leq 1\} = P\{X = 0\} + P\{X = 1\} =$

$$\binom{10}{0} p^0(1 - p)^{10} + \binom{10}{1} p^1(1 - p)^9 = (1 - p)^9(1 + 9p)$$

Test for values of p = 0.011, 0.012, 0.013, etc we have for p = 0.014

$(1 - 0.014)^9[1 + 9(1.014)] \approx 0.99$.

**11.**
Let X equal the number of chips found defective. We need to find p so that $P\{X \geq 2\} = 0.01$.

We know

$P\{X \geq 2\} = 1 - P\{X \leq 1\}$.

$P\{X \leq 1\} = 1 - P\{X \geq 2\} = 0.99$.

$P\{X \leq 1\} = P\{X = 0\} + P\{X = 1\} =$

$$\binom{n}{0}(0.05^0)(0.95)^{10} + \binom{n}{1}(0.05)^1(0.95)^9 = 0.95^{10} + n0.05(0.95)^9 = 0.99.$$

Solving the equation $n0.05(0.95)^9 + 0.95^{10} = 0.99$, gives $n \approx 12$.

**12.**

➤a.

$\#\mathbf{S} = N!$

$\#\{X_k = 1\} = (N - 1)!$

$$P\{X_k = 1\} = \frac{\#\{X_k = 1\}}{\#S} = \frac{(N-1)!}{N!} = \frac{1}{N}$$

➤b.

$$E(X_k) = 1\ P\{X_k = 1\} + 0\ P\{X_k = 0\} = \frac{1}{N}$$

➤c.

$$\sigma^2_{X_k} = E(X_k^2) - E(X_k)^2 = 1^2\ P\{X_k = 1^2\} + 0^2\ P\{X_k = 0^2\} - \frac{1}{N^2} = \frac{1}{N} - \frac{1}{N^2} = \frac{N-1}{N^2}$$

➤d.

The sequence does not have a binomial distribution because $X_k$ they are not independent.

$$P\{X_k \mid X_{k-1}\} = \frac{(N-2)!}{N!} = \frac{1}{N(N-1)}.$$

➤e.

S is equal to the number of matches.

➤f.

$$E(S) = E(X_1) + \ldots + E(X_N) = \frac{1}{N} + \frac{1}{N} + \ldots + \frac{1}{N} = \frac{N}{N} = 1$$

➤g.

$$\sigma_S^2 = E(S^2) - E(S)^2 = E(S^2) - 1$$

$$E(S^2) = E[(X_1 + \ldots + X_N)^2] = E(X_1^2) + E(X_2^2) + \ldots + E(X_N^2) +$$

$$E(X_1X_2) + E(X_1X_3) + \ldots + E(X_1X_N) +$$

$$E(X_2X_1) + E(X_2X_3) + \ldots + E(X_2X_N) + E(X_NX_1) + E(X_NX_2) + \ldots + E(X_NX_{N-1})$$

$E(X_k^2) = \dfrac{1}{N}$

$$E(X_1^2) + E(X_2^2) + ... + E(X_N^2) = \frac{1}{N} + \frac{1}{N} + ... + \frac{1}{N} = \frac{N}{N} = 1$$

For $p \neq q$,

$$E(X_pX_q) = (1)P(X_pX_q = 1) + (0)P(X_pX_q = 0) = \frac{(N-2)!}{N!} = \frac{1}{N(N-1)}$$

There are $N^2$ - N terms $E(X_p X_q)$. Therefore,

$$E(S^2) = 1 + (N^2 - N)\frac{1}{N(N-1)} = 2$$

$$\sigma_S^2 = E(S^2) - E(S)^2 = E(S^2) - 1 = 2 - 1 = 1$$

➤h.

$$E(\overline{P}) = E(\frac{S}{N}) = \frac{E(S)}{N} = \frac{1}{N}$$

$$\sigma^2_{\overline{P}} = E(\overline{P}^2) - E(\overline{P})^2 = E(\overline{P}^2) - \frac{1}{N^2} = E(\frac{S^2}{N^2}) - \frac{1}{N^2} = \frac{2}{N^2} - \frac{1}{N^2} = \frac{1}{N^2}$$

**13**.

➤a.

We can define a binomial experiment of N independent trials as follows:

Let $S = X_1 + X_2 + ... + X_N$ where

$\{X_k = 1\}$

is the event that on the kth trial success occurred and $\{X_k = 0\}$ failure occurs.

$\mu = E(S) = E(X_1 + X_2 + ... + X_N) = E(X_1) + E(X_2) + ... + E(X_N) =$

$1p + 0(1 - p) + 1p + 0(1 - p) + ... + 1p + 0(1 - p) = Np$.

➤b.

$E\{X_pX_q\} = 1P\{X_p = 1\ X_q = 1\} + 0P\{X_p Xq = 0\} = 1P\{X_p = 1\}P\{X_q = 1\} = p^2$

$E(X_k^2) = 1P\{X_k^2 = 1^2\} + 1P\{X_k^2 = 0^2\} = 1p$

$\sigma^2 = E(S^2) - E(S)^2 = E(S^2) - (Np)^2$

$E(S^2) = E[(X_1 + ... + X_N)^2] = E(X_1^2) + E(X_2^2) + ... + E(X_N^2) +$

$E(X_1X_2) + E(X_1X_3) + ... + E(X_1X_N) +$

$E(X_2X_1) + E(X_2X_3) + ... + E(X_2X_N) + E(X_NX_1) + E(X_NX_2) + ... + E(X_NX_{N-1}) =$

$Np + (N^2 - N)p^2$

$\sigma^2 = E(S^2) - E(S)^2 = E(S^2) - (Np)^2 = Np + (N^2 - N)p^2 - (Np)^2 = Np(1 - p)$

Therefore,

$\sigma = \sqrt{Np(1-p)}.$

**14.**
Since X, Y are independent for $0 \le k \le m + n$,

$P(X + Y = k) = P(X = k; Y = 0) + P(X = k - 1; Y = 1) + ... + P(X = 0; Y = k) =$

$P(X = k)P(Y = 0) + P(X = k - 1)P(Y = 1) + ... + P(X = 0)P(Y = k) =$

$$\binom{n}{k} p^k q^{n-k} \binom{m}{0} p^0 q^m + \binom{n}{k-1} p^{k-1} q^{n-k+1} \binom{m}{1} p^1 q^{m-1} +$$

$$\binom{n}{k-2} p^{k-2} q^{n-k+2} \binom{m}{2} p^2 q^{m-2} + ... + \binom{n}{0} p^0 q^n \binom{m}{k} p^k q^{m-k} =$$

$$\binom{n}{k}\binom{m}{0} p^k q^{n+m-k} + \binom{n}{k-1}\binom{m}{1} p^k q^{n+m-k} + ... + \binom{n}{0}\binom{m}{k} p^k q^{n+m-k} =$$

$$[\binom{n}{k}\binom{m}{0} + \binom{n}{k-1}\binom{m}{1} + ... + \binom{n}{0}\binom{m}{k}]p^k q^{n+m-k} = \binom{n+m}{k} p^k q^{n+m-k}$$

from Lesson 18, problem 8.

Therefore, Z = X + Y is a binomial distribution where,

$$P(Z = X + Y) = \binom{n+m}{k} p^k q^{n+m-k}$$

**15.**

➤a.

Here, N = 52, n = 13, r = 7.

We need to find $P(X \geq 1)$.

$$P(X \geq 1) = 1 - P(X = 0) = 1 - \frac{\binom{13}{0}\binom{52-13}{7-0}}{\binom{52}{7}} \approx 1 - 0.11 = 0.89.$$

➤b.

Here N = 52, n = 13, r = 7, p = 13/52 = 1/4, q = 3/4.

We need to find $P(X \geq 1)$.

$$P(X \geq 1) = 1 - P(X = 1) = 1 - \binom{7}{0}(\frac{1}{4})^0(3/4)^{7-0} = 1 - 0.13 = 0.87.$$

➤c.

0.89 - 0.87 = 0.02

Here, N = 52, n = 13, r = 7

We need to find $P(X \geq 1)$.

➤d.

Here, N = 260, n = 65, r = 7 and we need to find $P(X \geq 1)$.

$$P(X \geq 1) = 1 - P(X = 0) = 1 - \frac{\binom{65}{0}\binom{260-65}{7-0}}{\binom{260}{7}} \approx 1 - 0.13 = 0.87.$$

➤e.

Here N = 260, n = 65, r = 7, p = 65/260 = 1/4, q = 3/4. We need to find $P(X \geq 1)$.

$$P(X \geq 1) = 1 - P(X = 1) = 1 - \binom{7}{0}(\frac{1}{4})^0(3/4)^{7-0} = 1 - 0.13 = 0.87.$$

➤f.
0.87 - 0.87 = 0

➤g.
If sampling is done from a relatively large population, we do not have to be overly concerned about whether we have duplications or not in our sample even if the sample size is small.

**16.**

$$P(X = k) = \frac{\binom{4}{k}\binom{48}{5-k}}{\binom{52}{5}}$$

$$P(Y = k) = \frac{\binom{4}{k}\binom{48}{5-k}}{\binom{52}{5}}$$

Since X, Y are independent and $0 \le k \le 8$,

$$P(Z = X + Y = 0) = P(X = 0)P(Y = 0) = \binom{4}{0}\binom{48}{5}\binom{4}{0}\binom{48}{5}\binom{52}{5}^{-2} = \binom{48}{5}^2\binom{52}{5}^{-2}$$

P(Z = 1) = P(X = 1)P(Y = 0) + P(X = 0)P(Y = 1) =

$$\binom{4}{1}\binom{48}{4}\binom{4}{0}\binom{48}{5}\binom{52}{5}^{-2} + \binom{4}{0}\binom{48}{5}\binom{4}{1}\binom{48}{4}\binom{52}{5}^{-2}$$

P(Z = 2) =P(X = 2)P(Y = 0) + P(X = 1)P(X = 1) + P(X = 0)P(Y = 2) =

$$\binom{4}{2}\binom{48}{3}\binom{4}{0}\binom{48}{5}\binom{52}{5}^{-2} + \binom{4}{1}\binom{48}{4}\binom{4}{1}\binom{48}{4}\binom{52}{5}^{-2} + \binom{4}{0}\binom{48}{5}\binom{4}{2}\binom{48}{3}\binom{52}{5}^{-2}$$

P(Z = 3) = P(X = 3 )P(Y = 0) + P(X = 2)P(Y = 1) + P(X = 1)P(Y = 2) + P(X = 0 )P(Y = 3) =

$$\binom{4}{3}\binom{48}{2}\binom{4}{0}\binom{48}{5}\binom{52}{5}^{-2} + \binom{4}{2}\binom{48}{3}\binom{4}{1}\binom{48}{4}\binom{52}{5}^{-2} + \binom{4}{1}\binom{48}{4}\binom{4}{2}\binom{48}{3}\binom{52}{5}^{-2} +$$

$$\binom{4}{0}\binom{48}{5}\binom{4}{3}\binom{48}{2}\binom{52}{5}^{-2}$$

P(Z=4) = P(X = 4)(Y=0)+P(X=3)P(Y=1)+P(X = 2)P(Y =2)+P(X =1)P(Y =3)+P(X =0)P(Y = 4) =

$$\binom{4}{4}\binom{48}{1}\binom{4}{0}\binom{48}{5}\binom{52}{5}^{-2}+\binom{4}{3}\binom{48}{2}\binom{4}{1}\binom{48}{4}\binom{52}{5}^{-2}+\binom{4}{2}\binom{48}{3}\binom{4}{2}\binom{48}{3}\binom{52}{5}^{-2}$$

$$+\binom{4}{1}\binom{48}{4}\binom{4}{3}\binom{48}{2}\binom{52}{5}^{-2}+\binom{4}{0}\binom{48}{5}\binom{4}{4}\binom{48}{1}\binom{52}{5}^{-2}$$

P(Z = 5) = P(X = 4)P(Y = 1) + P(X = 3)P(Y = 2) + P(X = 2)P(Y = 3) P(X = 1)P(Y = 4) =

$$\binom{4}{4}\binom{48}{1}\binom{4}{1}\binom{48}{4}\binom{52}{5}^{-2}+\binom{4}{3}\binom{48}{2}\binom{4}{2}\binom{48}{3}\binom{52}{5}^{-2}+\binom{4}{2}\binom{48}{3}\binom{4}{3}\binom{48}{2}\binom{52}{5}^{-2}$$

P(Z = 6 ) = P(X = 4)P(Y = 2) + P(X = 3)P(Y = 3) + P(X = 2)P(Y = 4) =

$$\binom{4}{4}\binom{48}{1}\binom{4}{2}\binom{48}{3}\binom{52}{5}^{-2}+\binom{4}{3}\binom{48}{2}\binom{4}{3}\binom{48}{2}\binom{52}{5}^{-2}+\binom{4}{2}\binom{48}{3}\binom{4}{4}\binom{48}{2}\binom{52}{5}^{-2}$$

**17.**
X: The number of kings drawn from one deck of cards.
Y: The number of kings drawn from the other deck of cards.

Bayes Theorem:

P[X = 2; Y = 2|Z = X + Y = 4] = P[Z = X + Y = 4|X = 2; Y = 2]P(X = 2;Y = 2)/P(X + Y = 4)

Step 1: P[Z = X + Y = 4|X = 2; Y = 2] = 1

Therefore,

P[X = 2; Y = 2|Z = X + Y = 4] = P(X = 2;Y = 2)/P(X + Y = 4)

Step 2: P(X = 2;Y = 2) = P(X = 2)P(Y = 2) = $\dfrac{\binom{4}{2}\binom{48}{3}\binom{4}{2}\binom{48}{3}}{\binom{52}{5}\binom{52}{5}}$

Step 3: P(X + Y = 4) =

P(X = 4)(Y=0)+P(X=3)P(Y=1)+P(X = 2)P(Y =2)+P(X =1)P(Y =3)+P(X =0)P(Y = 4) =

$$\binom{4}{4}\binom{48}{1}\binom{4}{0}\binom{48}{5}\binom{52}{5}^{-2} + \binom{4}{3}\binom{48}{2}\binom{4}{1}\binom{48}{4}\binom{52}{5}^{-2} + \binom{4}{2}\binom{48}{3}\binom{4}{2}\binom{48}{3}\binom{52}{5}^{-2}$$

$$+ \binom{4}{1}\binom{48}{4}\binom{4}{3}\binom{48}{2}\binom{52}{5}^{-2} + \binom{4}{0}\binom{48}{5}\binom{4}{4}\binom{48}{1}\binom{52}{5}^{-2}$$

Step 4: P[X = 2; Y = 2|Z = X + Y = 4] = P(X = 2;Y = 2)/P(X + Y = 4) =

$$\frac{\binom{4}{2}\binom{48}{3}\binom{4}{2}\binom{48}{3}}{\binom{52}{5}\binom{52}{5}} / \text{P(X + Y = 4)} =$$

$$\binom{4}{2}\binom{48}{3}\binom{4}{2}\binom{48}{3}[\binom{4}{4}\binom{48}{1}\binom{4}{0}\binom{48}{5} + \binom{4}{3}\binom{48}{2}\binom{4}{1}\binom{48}{4} + \binom{4}{2}\binom{48}{3}\binom{4}{2}\binom{48}{3} +$$

$$\binom{4}{1}\binom{48}{4}\binom{4}{3}\binom{48}{2} + \binom{4}{0}\binom{48}{5}\binom{4}{4}\binom{48}{1}]^{-1} =$$

(36)(2991516160)/[ 1712304 + 3511779840 + 2991516160 + 3511779840 + 1712304]$^{-1}$ =

(36)(46742440)/313078139 ≈ 0.30

**18.**

P(X = k) = $\binom{N}{k} p_1^k q_1^{r-k}$ ,

$P(Y = k) = \binom{M}{k} p_2^k q_2^{r-k}.$

$P(X = n;Y = m) = P(X = n)P(Y = m) = \binom{N}{k} p_1^k q_1^{r-k} \binom{M}{k} p_2^k q_2^{r-k}$

$P(Z = X + Y = k) = P(X = 0;Y = k) + P(X = 1;Y = k - 1) + ... + P(X = k;Y = 0) =$

$P(X = 0)P(Y = k) + P(X = 1)P(Y = k - 1) + ... + P(X = k)P(Y = 0) =$

$\binom{N}{0}\binom{M}{k} p_1^0 q_1^N p_2^k q_2^{M-k} + \binom{N}{1}\binom{M}{k-1} p_1^1 q_1^{N-1} p_2^{k-1} q_2^{M-k+1} + ... +$

$\binom{N}{k}\binom{M}{0} p_1^k q_1^{N-k} p_2^0 q_2^M$

**19.**
machine A: $p_1 = 0.03$; $q_1 = 0.97$; $N_1 = 60$.
machine B: $p_2 = 0.05$; $q_2 = 0.95$; $N_2 = 40$.
machine C: $p_3 = 0.01$; $q_3 = 0.99$; $N_3 = 50$.
X: equals the number of defective parts for machine A.
Y: equals the number of defective parts for machine B.
Z: equals the number of defective parts for machine C.

$T = X + Y + Z$

$P(T \geq 2) = 1 - P(Y \leq 1) = 1 - P(T = 0) + P(T = 1)$

$P(T = 0) = P(X = 0;Y = 0; Z = 0) = P(X = 0)P(Y = 0)P(Z = 0) =$

$\binom{60}{0}(0.03^0)(0.97^{60})\binom{40}{0}(0.05^0)(0.95^{40})\binom{50}{0}(0.01^0)(0.99^{50}) \approx (0.16)(0.13)(0.61) = 0.01$

$P(T = 1) = P(X = 1)P(Y = 0)P(Z = 0) + P(X = 0)P(Y = 1)P(Z = 0) + P(X = 0)P(Y = 0)P(Z = 1) =$

$\binom{60}{1}(0.03^1)(0.97^{59})\binom{40}{0}(0.05^0)(0.95^{40})\binom{50}{0}(0.01^0)(0.99^{50})$ +

$\binom{60}{0}(0.03^0)(0.97^{60})\binom{40}{1}(0.05^1)(0.95^{39})\binom{50}{0}(0.01^0)(0.99^{50})$ +

$\binom{60}{0}(0.03^0)(0.97^{60})\binom{40}{0}(0.05^0)(0.95^{40})\binom{50}{1}(0.01^1)(0.99^{49}) \approx$

$(0.30)(0.13)(0.61) + (0.16)(0.27)(0.61) + (0.16)(0.13)(0.31) \approx 0.02 + 0.03 + 0.01 = 0.06$

$P(T \geq 2) = 1 - P(T \leq 1) = 1 - P(T = 0) + P(T = 1) = 1 - 0.01 - 0.06 = 0.93.$

**20.**

Step 1: **E**: The event that all the defective parts came from machine A.

**E** $= [(X > 0)\cap(Y = 0)\cap(Z = 0]$

$P\{$**E**$|T \geq 2) = P[(X > 0)\cap(Y = 0)\cap(Z = 0)|T \geq 2]$

Step 2: From Baye's theorem: $P[(X > 0)\cap(Y = 0)\cap(Z = 0)|T \geq 2] =$

$P[T \geq 2|(X > 0)\cap(Y = 0)\cap(Z = 0)]P[(X > 0)\cap(Y = 0)\cap(Z = 0)]/P(T \geq 2) =$

$P[X \geq 2|(X > 0)\cap(Y = 0)\cap(Z = 0)]P[(X > 0)\cap(Y = 0)\cap(Z = 0)]/P(T \geq 2)$

Step 3: $P[X \geq 2|(X > 0)\cap(Y = 0)\cap(Z = 0)] =$

$P\{(X \geq 2)\cap(Y = 0)\cap(Z = 0)\}/P\{(X > 0)\cap(Y = 0)\cap(Z = 0)\} =$

$[P(X \geq 2)P(Y = 0)P(Z = 0)]/[P(X > 0)P(Y = 0)P(Z = 0)] = P(X \geq 2)/P(X > 0)$

Step 4: $P(X \geq 2) = 1 - P(X = 0) - P(X = 1) = 1 - \binom{60}{0}(0.03^0)(0.97^{60}) - \binom{60}{1}(0.03)(0.97^{59})$

$\approx 1 - 0.16 - 0.30 = 0.54$

$P(X > 0) = 1 - P(X = 0) = 1 - 0.16 = 0.84$

$P[T \geq 2|(X > 0)\cap(Y = 0)\cap(Z = 0)] = P(X \geq 2)/P(X > 0) = 0.54/0.84 \approx 0.64$

Step 5: $P[(X > 0)\cap(Y = 0)\cap(Z = 0)] = P(X > 0)P(Y = 0)P(Z = 0) = (0.84)(0.95^{40})(0.99^{50}) \approx 0.07$

Step 6: From problem 19, $P(T \geq 2) = 0.93$ and Bayes:

$P[(X > 0)\cap(Y = 0)\cap(Z = 0)|T \geq 2] =$

$P[T \geq 2|(X > 0)\cap(Y = 0)\cap(Z = 0)]P[(X > 0)\cap(Y=0)\cap(Z = 0)]/P(T \geq 2) = (0.64)(0.07)/(0.93) \approx 0.05$

**21.**

Since $P(X = k) = P(Y = k) = \binom{r}{k} p^k q^{r-k}$ and from problem 14,

$$P(X + Y = N) = \binom{r+r}{N} p^N q^{r+r-N} = \binom{2r}{N} p^N q^{2r-N}$$

$P(X = k|X + Y = N) = P\{(X = k)\cap(X + Y = N)\}/P(X + Y = N) =$

$P\{(X = k)\cap(Y = N - k)\}/P(X + Y = N) = P(X = k)P(Y = N - k)/P(X + Y = N) =$

$$\binom{r}{k} p^k q^{r-k} \binom{r}{N-k} p^{N-k} q^{r-N+k} \Big/ \binom{2r}{N} p^N q^{2r-N} = \frac{\binom{r}{k}\binom{r}{N-k}}{\binom{2r}{N}}$$

which is hypergeometric distribution.

**22.**

Here, $N = 52$, $n = 13$, $r$ = number of cards drawn and we need to find $P(X \geq 2)$.

$$P(X \geq 2) = 1 - P(X = 0) - P(X = 1) = 1 - \frac{\binom{13}{0}\binom{52-13}{r-0}}{\binom{52}{r}} - \frac{\binom{13}{1}\binom{52-13}{r-1}}{\binom{52}{r}} = \geq 0.90.$$

From problem 14, For large N, we have

$$\frac{\binom{n}{k}\binom{N-n}{r-k}}{\binom{N}{r}} \approx \binom{r}{k} p^k q^{r-k}.$$

Therefore,

$$P(X \geq 2) \approx 1 - \binom{r}{0}(1/4)^0(3/4)^r - \binom{r}{1}(1/4)^1(3/4)^{r-1} = 1 - (3/4)^r - r\,(1/4)(3/4)^{r-1} \geq 0.90$$

$$(3/4)^r + r\,(1/4)(3/4)^{r-1} \leq 0.10$$

Testing for r ranging from $1 \leq r \leq 14$, we find that r = 14.

Testing the hypergeometric distribution we have

$$P(X \geq 2) = 1 - \frac{\binom{13}{0}\binom{52-13}{14-0}}{\binom{52}{14}} - \frac{\binom{13}{1}\binom{52-13}{14-1}}{\binom{52}{14}} = 1 - \frac{\binom{39}{14}}{\binom{52}{14}} - \frac{13\binom{39}{13}}{\binom{52}{14}} \approx 0.93$$

Therefore r = 14 is a good estimate.

**23.**

As problem 22, shows, we can use the binomial distribution as a good approximation where

r= sample size, p = 0.10, q = 0.95

X: The r.v. equal to the number of math majors selected in the sample.

$$P(X = k) = \binom{r}{k} 0.05^k \; 0.95^{r-k}$$

$P(X \geq 10) = 1 - P(X \leq 9) \geq 0.90$

It follows that $P(X \leq 9) \leq 0.10$ and

$P(X\leq 9) = P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)+P(X=5)+P(X=6)+P(X=7)+P(X=8)+P(X=9)$

Using a calculator and trial and error, we find the best approximation is r = 280

where

$P(X = 9) \approx 0.046$
$P(X = 8) \approx 0.029$
$P(X = 7) \approx 0.016$

$P(X = 6) \approx 0.008$

$P(X = 5) \approx 0.003$

$P(X = 4) \approx 0.001$

$P(X = 3) \approx 0.0$

$P(X = 2) \approx 0.0$

$P(X = 1) \approx 0.0$

$P(X = 0) \approx 0.0$

Summing these numbers we have $P(X \leq 9) \approx 0.10$

Therefore, for a sample size of r = 280, $P(X \geq 10) \approx 0.90$

**24.**

Here p = 1/2, q = 1/2, r = 5, n = 10

$$P(X = 10) = \binom{10-1}{5-1}(\frac{1}{2})^5(\frac{1}{2})^{10-5} \approx 0.123$$

**25.**

➤a.

Here p = 0.05, q = 0.95, r = 2, n = 10

$$P(X = 10) = \binom{10-1}{2-1}(0.05)^2(0.95)^{10-2} \approx 0.02$$

➤b.

Here p = 0.05, q = 0.95, r = 2, n = 20

$$P(X = 20) = \binom{20-1}{2-1}(0.05)^2(0.95)^{20-2} \approx 0.02$$

**26.**

➤a.

Let X be the r.v. that equals the number of tosses of the coin:

X = 5,6,...,10

For n < 10, X will have a Pascal distribution where

r = 5, p = 1/2, q = 1/2, N = 5,6,7,8,9.

Therefore we have

$$P(X = 5) = \binom{5-1}{5-1}\left(\frac{1}{2}\right)^5 \approx 0.031$$

$$P(X = 6) = \binom{6-1}{5-1}\left(\frac{1}{2}\right)^6 \approx 0.078$$

$$P(X = 7) = \binom{7-1}{5-1}\left(\frac{1}{2}\right)^7 \approx 0.117$$

$$P(X = 8) = \binom{8-1}{5-1}\left(\frac{1}{2}\right)^8 \approx 0.137$$

$$P(X = 9) = \binom{9-1}{5-1}\left(\frac{1}{2}\right)^9 \approx 0.137$$

$$P(X = 10) \approx 1 - (0.031 + 0.078 + 0.117 + 0.137 + 0.137) = 1 - 0.5 = 0.5$$

$$E(X) = 5(0.031) + 6(0.078) + 7(0.117) + 8(0.139) + 9(0.137) + 10(0.5) \approx 8.79$$

➤b.
Step 1: We first compute

$$P(X = 10) = \binom{10-1}{5-1}\left(\frac{1}{2}\right)^{10} \approx 0.123$$

Step 2: From a. we have

$$P(5 \le X \le 10) = (0.031 + 0.078 + 0.117 + 0.137 + 0.137) + 0.123 = 0.623$$

Since the game must end by the $10^{th}$ toss, the probability that 5 heads will not occur equals

$$1 - P(5 \le X \le 10) = 1 - 0.623 = 0.377.$$

**27.**
The sample space generated by the Pascal random variable will have an infinite cardinality since the number of trials n can be arbitrary large. However, for the sample space generated by a binomial random variable, the number of trials is a given n. And therefore the cardinality of the sample space will be $2^n$.