Probability theory
Lesson 18
The Binomial Sample Space

18.1 - What is the Binomial Experiment that Generates a Special Sample Space?

18.1 - Problem 1:
➤(a).
It is reasonable to assume the selection of each of the three stores independent choices.

Let m be the occurrence of a store having a million dollars in sales.
Let f be the occurrence of a store not having a million dollars in sales. Since we are selecting three stores, the sample space consists of eight possibilities:

S = {(m,m,m),(m,m,f),(m,f,m),(f,m,m),(m,f,f),(f,m,f),(f,m,m),(f,f,f)}

➤(b).
We select all the elements of **S** that consists of three ms' and one f:

E = {(f,m,m),(m,f,m),(m,m,f)}

➤(c).
1. Since there are a large number of stores to sample from and the sample size is relatively small, it is reasonable to assume the sampling is independent.

2. Each sample results in only two possible outcomes: a store has or does not have a million dollars in sale. We define success as a store that has a million dollars in sales.

3. The probability of success for each sample is $p = 0.20$ which is the proportion of stores that are estimated to have a million dollars in sales. The probability of failure is $1-0.20 = 0.80$.

18. 1 - Problem 2:
➤(a).
Since there is such a large number of students in her classes, it is reasonable to assume the selections are independent of each other.

Let s indicate that a student selected has a grade of B or better.
Let f indicate failure: that a student selected has a grade of C or less.
The sample space are the following eight possibilities:

S = {(s,s,s),(s,s,f),(s,f,s),(f,s,s),(f,f,s),(f,s,f),(s,f,f),(f,f,f)}

➤(b).
We write out all possible ways for two s and one f to occur:

E = {(s,s,f),(s,f,s),(f,s,s)}

➤(c).
Step 1: Since she has a large number of students and the sample is relatively small, it is reasonable to assume that the sampling results in independent selections.

Step 2: We define success as a student selected that has a grade of B or better.

Step 3: The probability for success, p = 0.45 and the probability for failure is 0.55.

Problem 1.3:
➤(a).
It is reasonable to assume the independence of hiring the 3 people.

Let l indicate that a trainee has studied international law.

Let f indicate that a trainee has not studied international law.

S = {(l,l,l),(l,l,f),(l,f,l),(f,l,l),(f,f,l),(f,l,f),(l,f,f),(f,f,f)}

➤(b).
E = {(f,f,f)}

Since we are interested in the event that none of the trainees have studied international law, we have **E** = {(f,f,f)}.

➤(c).
1. Each trainee is considered an independent trial. The number of trials is 3.

2. Each trial results in success that a trainee has studied international law or failure that a trainee has not studied international law.

3. The probability for success is p that a trainee has studied international law and the probability is q = 1 - p that the a trainee has not studied international law.

18.1 - Problem 4:
➤(a).
Let m indicate a book is a murder mystery and f indicate a book is not a murder mystery. Since we are only selecting 2 books, we have **S** = {(m,m),(m,f),(f,m),(f,f)}.

➤(b).
Since both books are murder mysteries, we have

E = {(m,m)}

➤(c).
1. Each drawing of a book is a dependent trial rather than independent.

2. For each random selection of a book, the probability of selecting a murder mystery changes. For example, if there are only two mystery books in his library, then

$P(M_1 \cap M_2)) = (\frac{2}{30})(\frac{1}{29})$,

where $M_1 \cap M_2$ is the event that the first book is a mystery book and the second book is a mystery book .

18.2 - Counting the Number of Successes from a Binomial Experiment.

18.2 - Problem 1:
Method 1: Write out the event **E** that exactly two are girls:

E = {(g,g,b),(g,b,g),(b,g,g)}

#**E** = 3.

Method 2: Think of the gender of each child s filling up each of the three slot positions:

_________, ________, _______. Since we are interested in two girls, then two slots are reserved for girls. Therefore, the number of ways of getting 2 girls is equal to the number of ways of selecting 2 girls out of 3 girls where order is not important. Using the counting methods of lesson 8, we have

#**E** = $\binom{3}{2}$ = 3.

18.2 - Problem 2:
➤(a)
Think of the bullets that hit the target filling up each of the six slot positions:

_________, ________, _______,_________, ________, _______.

Since we are interested in three bullets hitting the target, then three slots are reserved for bullets that hit the target. Therefore, the number of ways of that 3 bullets hit the target is equal to the number of ways of selecting 3 slots out of 6 slots where order is not important. Using the counting methods of lesson 8, we have

#**E** = $\binom{6}{3}$ = 20 ways three bullets hit the target.

➤(b)
Since we are interested in four bullets <u>not</u> hitting the target, then four slots are reserved for bullets that do not hit the target. Therefore, the number of ways that 4 bullets do not hit the target is equal to the number of ways of selecting 4 slots out of 6 slots where order is not important. Using the counting methods of lesson 8,

#**E** = $\binom{6}{4}$ = 15 ways four bullets miss the target.

Supplementary Problems

1.
In each toss of the die, if a 2,4,5,6 occurs, we define these numbers by the symbol f, meaning a failure has occurred. Therefore, the follow is the sample space:

S = {(1,1),(1,f),(f,1),(3,3),(3,f),(f,3),(3,1),(1,3),(f,f)}

2.
Step 1: In each toss of the die, if a 2,4,6 occurs we define these numbers by the symbol f, meaning a failure has occurred.

Step 2: For each toss of the die, there is 4 possible outcomes: 1,3,5, and f.

Step 3: Since the are N tosses, the number of possibilities for the sample space is #**S** = 4^N.

3.
In writing out the event **E** we need to consider all possible ways a or 3 appears in 4 tosses of the die. Therefore,

E = {(1,1,3,3),(1,3,1,3),(3,1,3,1),(1,3,3,1),(3,3,1,1),(3,1,1,3)}

4.

Step 1: The number of ways the event e_1 can occur out of N trials is $\binom{N}{r_1}$

leaving N $-r_1$ trials to occur.

Step 2: The number of ways the event e_2 can occur out of N $-r_1$ trials is

$$\binom{N - r_1}{r_2}$$ leaving N $-r_1 - r_2$ trials to occur.

Step 3: The number of ways the event e_3 can occur out of N $-r_1 - r_2$ trials is

$$\binom{N - r_1 - r_2}{r_3}$$ leaving N $-r_1 - r_2 - r_3$ trials to occur.

....

Step n: The number of ways e_n can occur out of N $- r_1 - r_2 - ... - r_{n-1}$ trials is

$$\binom{N - r_1 - r_2 - r_3 - ... - r_{n-1}}{r_n} .$$

The product will give the following formula for the total number of ways the events e_i, i = 1,...n can occur in N trials:

$$\binom{N}{r_1}\binom{N-r_1}{r_2}\binom{N-r_1-r_2}{r_3}\binom{N-r_1-r_2-r_3}{r_4}\binom{N-r_1-r_2-r_3-r_4}{r_5}...\binom{N-r_1-r_2-r_3-r_4-...-r_{n-1}}{r_n} =$$

$$\frac{N!}{r_1!r_2!r_3!r_4!...r_n!}$$

5.

Step 1 :We will use the multinomial formula in problem 4 where N = 10, n = 4 and

e_1: 11 occurs two times,

e_2: 10 occurs three times,

e_3: 4 occurs seven times,

e_4: any other number can occur one time,

Step 2: Using the mulinomial formula:

$$\frac{10!}{2!3!4!1!} = \frac{(10)(9)(8)(7)(6)(5)}{(2)(6)} = (5)(9)(8)(7)(5) = 12{,}600.$$

6.
Step 1 :We will use the multinomial formula in problem 4 where N = 10, n = 3 and
e_1: wins 5 games,
e_2: losses 3 games,
e_3: ties 2 games.

Step 2: Using the mulinomial formula:

$$\frac{10!}{5!3!2!} = \frac{(10)(9)(8)(7)(6)}{(3!)(2!)} = 2{,}520.$$

7.
We cannot use the multinomial formula in problem 4 because order is not important for a hand of cards.

The number of ways of selecting 2 aces out of 4 aces is

$$\binom{4}{2} = 6.$$

The number of ways of selecting 2 kings out of 4 kings is

$$\binom{4}{2} = 6.$$

The number of ways of selecting 1 jack out of 4 jacks is 4.

The number of ways of selecting 2 of the other cards is

$$\binom{40}{2} = 780.$$

Therefore, the number of ways of getting such a hand is (6)(6)(4)(780) = 112,320.

8.
From algebra, we have the binomial formula:

Step 1:

$$(a + b)^N = \binom{N}{0} a^N b^0 + \binom{N}{1} a^{N-1} b^1 + \ldots + \binom{N}{k} a^{N-k} b^k + \ldots + \binom{N}{N} a^0 b^N$$

$$(1+t)^N = \binom{N}{0} t^0 + \binom{N}{1} t^1 + \ldots + \binom{N}{k} t^k + \ldots + \binom{N}{N} t^N$$

Step 2: Define $a_k = \binom{n}{k}$ and $b_k = \binom{m}{k}$

$(1 + t)^n = a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n$

$(1 + t)^m = b_0 + b_1 t + b_2 t^2 + \ldots + b_m t^m$

$(1 + t)^n(1 + t)^m = a_0b_0 + (a_0b_1 + a_1b_0)t + (a_0b_2 + a_1b_1 + a_2b_0)t^2 + \ldots + (a_0b_k + a_1b_{k-1} + \ldots + a_kb_0)t^k + \ldots$

$(1 + t)^{n+m} = c_0 + c_1 t + c_2 t^2 + \ldots + c_{n+m} t^{n+m}$

Since $a_k = \binom{n}{k} = 0$ for $k > n$ we can match the coefficients as following:

$c_{k\,=} (a_0b_k + a_1b_{k-1} + \ldots + a_kb_0)$

which gives

$$c_k = \binom{m+n}{k} = \binom{n}{0}\binom{m}{k} + \binom{n}{1}\binom{m}{k-1} + \ldots . \binom{n}{k}\binom{m}{0}$$

9.

Step 1:

$P(X \le k) = p + q^2p + \ldots + q^{k-1}p = p(1 + q^2 + \ldots + q^{k-1}) = p(1 + q^2 + \ldots + q^{k-1})(1 - q)/(1 - q) =$

$p(1 + q^2 + \ldots + q^{k-1})(1 - q)/(p) = (1 + q^2 + \ldots + q^{k-1})(1 - q) = 1 - q^k$

Step 2: $P(X \le k) + P(X > k) = 1$

$P(X > k) = 1 - P(X \le k) = 1 - (1 - q^k) = q^k$