PROBABILITY THEORY
Lesson 17
Factorials & Counting Samples

17.1- What are Factorials?

17.1 - Problem 1:
We know 10! = 3,628,800

Therefore, 13! = (13)(12)(11)(10!) = (13)(12)(11)(3,628,800) = 6,227,020,800

17.1 - Problem 2:

$$\frac{77!}{74!} = \frac{(77)(76)(75)(74!)}{74!} = (77)(76)(75) = 438{,}900$$

17.1 - Problem 3:
(10!)(10!) = (3,628,800)(3,628,800) = 13,168,189,440,000

17.1 - Problem 4:

$$\frac{20!}{10!10!} = \frac{(20)(19)(18)(17)(16)(15)(14)(13)(12)(11)(10!)}{(10!)(10!)} =$$

$$\frac{(20)(19)(18)(17)(16)(15)(14)(13)(12)(11)}{(10)(9)(8)(7)(6)(5)(4)(3)(2)} = (19)(2)(17)(13)(2)(11) = 184{,}756$$

17.2 - Counting the Number of Samples

17.2 - Problem 1:
Since two different prizes are rewarded, order is important. We use the formula

$$_{N}P_{r} = \frac{N!}{(N - r)!}$$

N = 25 and r = 2.

Therefore,

$$_{25}P_2 = \frac{25!}{(25 - 2)!} = \frac{25!}{23\,!} = (25)(24) = 600\ .$$

17.2 - Problem 2:
For this problem order is important.

Step 1: The number of ways the 3 girls out of 7 girls can stand together is

$$_7\mathbf{P}_3 = \frac{7!}{(7 - 3)!} = \frac{7!}{4\,!} = (7)(6)(5) = 210.$$

Step 2: The number of ways the 3 boys out of 5 boys can stand together is

$$_5P_3 = \frac{5!}{(5 - 3)!} = \frac{5!}{2\,!} = (5)(4)(3) = 60.$$

Step 3: Assume all the girls stand on the left of the boys. The number of ways the girls and boys can stand in a row is

(210)(60) = 12,600

Step 4: Assume all the girls stand on the right of the boys. The number of ways the girls and boys can stand in a row is

(210)(60) = 12,600.

Step 5: Therefore, the number of ways girls can stand next to the boys is

(2)(210)(60) = (2)(12,600) = 25,200.

17.2 - Problem 3:
Step 1: Think of a list of 20 items where each item represents a student that approves or does not approve the President's foreign policy. From this list, we will assume 10 do not approve. We need to find the total number of possible ways this can happen where order is not important. To compute this number we use the formula

$$\binom{N}{r} = \frac{N!}{r!(N - r)!} \text{ where } N = 20 \text{ and } r = 10.$$

Therefore,

$$\binom{20}{10} = \frac{20!}{10!(20 - 10)!} = \frac{(20)(19)(18)(17)(16)(15)(14)(13)(12)(11)}{10!} =$$

$$\frac{(20)(19)(18)(17)(16)(15)(14)(13)(12)(11)}{(10)(9)(8)(7)(6)(5)(4)(3)(2)} = (19)(2)(17)(13)(2)(11) = 184{,}756.$$

17.2 - Problem 4:
Step 1: Since order is not important, we use the formula

$$\binom{N}{r} = \frac{N!}{r!(N - r)!}$$

Step 2: The number ways of selecting 4 basketball games out of 8 basketball games is

$$\binom{8}{4} = \frac{8!}{4!(8 - 4)!} = \frac{(8)(7)(6)(5)}{(4)(3)(2)} = (2)(7)(5) = 70.$$

Step 3: The number ways of selecting 5 football games out of 7 football games is

$$\binom{7}{5} = \frac{7!}{5!(7 - 5)!} = \frac{(7)(6)}{2} = 21.$$

Step 4: The number ways of selecting 5 soccer games out of 10 soccer games is

$$\binom{10}{5} = \frac{10!}{5!(10 - 5)!} = \frac{(10)(9)(8)(7)(6)}{(5)(4)(3)(2)} = (2)(9)(2)(7) = 252.$$

Step 5: The number of ways is (70)(21)(252) = 370,440.

17.3 - Probability Applications

17.3 - Problem 1:
The sample space is all possible ways the voters can stand in line:

#**S** = 10! = 3,628,800

E: The event that voters of the same party are standing together.

Step 1: The number of ways all Republicans can stand together is 3! = 6.

Step 2: The number of ways all Democrats can stand together is 2! = 2.

Step 3: The number of ways all Independents can stand together is 3! = 6.

Step 4: The number of ways all Libertarians can stand together is 2! = 2.

Step 4: However, each category can be arranged in 4! = 24 different ways.

Step 5: From the above steps,

#**E** = (24)(3!)(2!)(3!)(2!) = (24)(6)(2)(6)(2) = 3456.

Step 6: $P(\mathbf{E}) = \dfrac{\#\mathbf{E}}{\#\mathbf{S}} = \dfrac{3{,}456}{3{,}628{,}800}$.

17.3 - Problem 2:
There are 15 members on this committee. The sample space **S** is all possible ways of selecting from this committee 3 Democrats, 2 Republicans and 4 Independent members; a total of 9 members.

The number of ways possible is

$$\#\mathbf{S} = \binom{15}{9} = \frac{15!}{6!9!} = \frac{(15)(14)(13)(12)(11)(10)}{(6)(5)(4)(3)(2)} = (7)(13)(11)(5) = 5{,}005$$

Step 1: The number of ways of selecting 3 Democrats is $\binom{7}{3} = \dfrac{(7)(6)(5)}{(3)(2)} = 35.$

Step 2: The number of ways of selecting 2 Republicans is $\binom{3}{2} = 3.$

Step 3: The number of ways of selecting 4 Independents is $\binom{5}{4} = 5.$

Step 4: #**E** = (35)(3)(5) = 525.

Step 4: $P(\mathbf{E}) = \dfrac{525}{5005}$ **.**

Supplementary Problems

1..
The number of possible kings selected range from zero to five. Let k represent the number of kings selected from drawing 5 cards at random, without replacement.

Step 1: The number of ways of drawing k kings is

$$\binom{4}{k}\ .$$

Step 2: The number of ways of drawing the remaining non-kings is

$$\binom{48}{5-k}$$

Step 3: The number of ways of drawing 5 cards from an ordinary deck is

$$\binom{52}{5} = \frac{52!}{5!47!} = \frac{(52)(51)(50)(49)(48)}{(5)(4)(3)(2)} = (52)(51)(10)(49)(2) = 2{,}598{,}960.$$

Step 4: Let X equal the number of kings drawn. Then

$$P\{X = k\} = \frac{\binom{4}{k}\binom{48}{5-k}}{\binom{52}{5}}$$

$$P\{X = 0\} = \frac{\binom{4}{0}\binom{48}{5}}{\binom{52}{5}} = \frac{\frac{48!}{5!43!}}{\binom{52}{5}} = \frac{(48)(47)(23)(3)(11)}{\binom{52}{5}} = \frac{1712304}{2598960}$$

$$P\{X = 1\} = \frac{\binom{4}{1}\binom{48}{4}}{\binom{52}{5}} = \frac{\frac{(4)48!}{4!44!}}{\binom{52}{5}} = \frac{(16)(47)(23)(45)}{\binom{52}{5}} = \frac{778320}{2598960}$$

$$P\{X=2\} = \frac{\binom{4}{2}\binom{48}{3}}{\binom{52}{5}} = \frac{\frac{48!}{45!}}{\binom{52}{5}} = \frac{(48)(47)(46)}{\binom{52}{5}} = \frac{103776}{2598960}$$

$$P\{X=3\} = \frac{\binom{4}{3}\binom{48}{2}}{\binom{52}{5}} = \frac{\frac{(4)48!}{(2!)46!}}{\binom{52}{5}} = \frac{(2)(48)(47)}{\binom{52}{5}} = \frac{4512}{2598960}$$

$$P\{X=4\} = \frac{\binom{4}{4}\binom{48}{1}}{\binom{52}{5}} = \frac{48}{\binom{52}{5}} = \frac{48}{2598960}$$

2.
Step 1: The number of ways that 5 girls out of 12 girls can stand together is

$${}_{12}P_5 = \frac{12!}{(12-5)!} = (12)(11)(10)(9)(8) = 95040.$$

Step 2: The number of ways that 5 boys out of 15 boys can stand together is

$${}_{15}P_5 = \frac{15!}{(15-5)!} = (15)(14)(13)(12)(11) = 360360.$$

Step 3: For the following sequences show how all the five boys selected can stand together where b stand for a boy and g a girl:

bbbbbggggg, gbbbbbgggg, ggbbbbbggg, gggbbbbbgg, ggggbbbbb, bbbbbggggg.

Therefore, the total number of ways the five boys can stand together is

$6({}_{12}P_5)({}_{15}P_5) = 6(95{,}040)(360{,}360)$

3.
Step 1: The number of ways that 4 girls can stand together is 4! = 24.

Step 2: The number of ways that 6 boys can stand together is 6! = 720.

Step 3: For the following sequences show how all the six boys selected can stand together where b stand for a boy and g a girl:

bbbbbbgggg, gbbbbbbggg, ggbbbbbbgg, gggbbbbbbg, ggggbbbbbb.

Therefore, the total number of ways the six boys can stand together is

5(24)(720) = 86400.

Step 4: The total number of ways her children can stand together is

10! = 3,628,800.

Step 5: The probability that all the boys are standing together is

$$\frac{86{,}400}{3{,}628{,}800} .$$

4.

➤a.

Step 1: For the natural order of cards we have

1 2 3 4 5 6 7 8 9 10

Step 2: Assume the second deck has a match on the first card. Since the remaining nine cards can be of any random order, the number of ways a match occurs on the first card is 9!.

Step 3: The total number of possible random shuffles is 10!

Step 4: The probability of a match on the first card is

$$\frac{9!}{10!} = \frac{1}{10} .$$

➤b.

Step 1: For the natural order of cards we have

1 2 3 4 5 6 7 8 9 10

Step 2: Assume the second deck has a match on the first and fifth card. Since the remaining eight cards can be of any random order, the number of ways a match occurs on the first card is 8!.

Step 3: The total number of possible random shuffles is 10!

Step 4: The probability of a match on the first card is $\frac{8!}{10!} = \frac{1}{90}$.

5.

➤a.

Step 1: The number of ways that 5 men can be seated together is 5! = 120.

Step 2: The number of ways that 5 women can be seated together is 5! = 120.

Step 3: For the following sequences show how all the five men can be seated together:

mmmmmwwwww, wmmmmmwwww, wwmmmmmwww, wwwmmmmmww, wwwwmmmmmw, wwwwwmmmmm .

Therefore, the total number of ways the five men can sit together is

6(120(120) = 86400.

Step 4: The total number of ways ten people can sit together is

10! = 3,628,800.

Step 5: The probability that all the men are seated together is

$$\frac{86{,}400}{3{,}628{,}800}\ .$$

➤b.

Since there are 5 couples, the number of ways the couples can be seated together is $(5!)2^5 = 3840$. Therefore, the probability that he couples will be seated together is

$$\frac{3840}{10!}.$$

6.

Step 1: Since each person can be born on any of the 12 months, the number of ways this can happen for the five people is

$(12)(12)(12)(12)(12) = 12^5$.

Step 2: The number of ways none are born on the same month is reasoned as follows:

Number each person 1 to 5.

Person number 1 has 12 choices for the month born.

Person number 2 has 11 choices for the month born.

Person number 3 has 10 choices for the month born.

Person number 4 has 9 choices for the month born.

Person number 5 has 8 choices for the month born.

Therefore, the number of ways the 5 people will not be born on the same month is

(12)(11)(10)(9)(8) = 95,040.

The probability that none will be born on the same month is

$$\frac{95040}{12^5}\ .$$

7.

tep 1: Let **A** be the event that the hand contains 3 aces. The number of ways this can happen is

$$\#\mathbf{A} = \binom{4}{3}\binom{48}{2} = (4)(1128) = 4512.$$

Step 2: Let **K** be the event that the hand contains 2 kings.

The number of ways this can happen is

$$\#\mathbf{K} = \binom{4}{2}\binom{48}{3} = (6)(103776) = 622656.$$

Step 3: Let **E** be the event that the hand contains 3 aces or 2 kings.

$\mathbf{E} = \mathbf{A} \cup \mathbf{K}$

Step 4: $P(\mathbf{E}) = P(\mathbf{A} \cup \mathbf{K}) = P(\mathbf{A}) + P(\mathbf{K}) - P(\mathbf{A} \cap \mathbf{K})$

Step 5: $\mathbf{A} \cap \mathbf{K}$ is the event that the hand contains 3 aces and 2 kings.

The number of ways this can happen is

#(**A**∩**K**) = $\binom{4}{3}\binom{4}{2}$ = (4)(6) = 24.

Step 6: The number of possible hands is $\binom{52}{5}$

Step 7: P(**E**) = P(**A**∪**K**) = P(**A**) + P(**K**) - P(**A**∩**K**) = $\dfrac{\binom{4}{3}\binom{48}{2} + \binom{4}{2}\binom{48}{3} - \binom{4}{3}\binom{4}{2}}{\binom{52}{5}}$.

8.
Step 1:
K: The event that two cards drawn are kings.
$\mathbf{K_S}$: The event that two cards are kings and one of these kings is a spade.
$\mathbf{K_N}$: The event that two cards are kings and none of these kings is a spade.

$\mathbf{K = K_S \cup K_N}$

B: The event that three spaces are drawn.

E: The event that two cards drawn are kings and three cards drawn are spades.

$\mathbf{E} = \mathbf{K}\cap\mathbf{B} = (\mathbf{K_S}\cup\mathbf{K_N})\cap\mathbf{B} = (\mathbf{K_S}\cap\mathbf{B})\cup(\mathbf{K_N}\cap\mathbf{B})$.

Step 2: #($\mathbf{K_S}\cap\mathbf{B}$) = $\binom{3}{1}\binom{12}{3}$

Step 3: #($\mathbf{K_N}\cap\mathbf{B}$) = $\binom{3}{2}\binom{12}{3}$

Step 4: #**E** = #($\mathbf{K_S}\cap\mathbf{B}$) + #($\mathbf{K_N}\cap\mathbf{B}$) = $\binom{3}{1}\binom{12}{3} + \binom{3}{2}\binom{12}{3}$

Step 5 P(**E**) = $\dfrac{\binom{3}{1}\binom{12}{3} + \binom{3}{2}\binom{12}{3}}{\binom{52}{5}}$.

9.
N = 7.

$$2^7 = \binom{7}{0} + \binom{7}{1} + \binom{7}{2} + \binom{7}{3} + \binom{7}{4} + \binom{7}{5} + \binom{7}{6} + \binom{7}{7} =$$

$1 + 7 + 21 + 35 + 35 + 21 + 7 + 1 = 128.$

10.

The number of single wages is $\binom{5}{1}$.

The number of 2 team parleys is $\binom{5}{2}$.

The number of 3 team parleys is $\binom{5}{3}$.

The number of 4 team parleys is. $\binom{5}{4}$

The number of 5 team parleys is $\binom{5}{5}$.

Therefore, the number of wages is $\binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5} = 2^5 - 1 = 31.$

11.
The number of disjoint subsets in the Venn diagram is 8. Therefore, there are $2^8 - 1 = 255$ subsets that can be generated.

12.
➤a.

Since there are 7 cards in the hand, there are $\binom{7}{3} = 35$ different ways for 3 kings to appear among the 7 cards, where order is NOT important. Of the remaining 4 cards there are $\binom{4}{2} = 6$**.**

Therefore, the total number of possible different ways 3 kings and 2 queens can happen is

$35(6) = 210.$

➤b.
Start with any arbitrary arrangement of 3 kings and 2 queens and 2 other cards other than kings or queens. The probability of such an arrangement is

$$(210)\frac{(4)}{(52)}\frac{(3)}{(51)}\frac{(2)}{(50)}\frac{(4)}{(49)}\frac{(3)}{(48)}\frac{(44)}{(47)}\frac{(43)}{(46)} = \frac{99{,}330}{585{,}307{,}450}.$$

13.

➤a.

Since to win order is not important, the total number of ways 6 different numbers can be selected from 51 numbers is

$$\binom{51}{6} = \frac{51!}{6!45!} = 18{,}009{,}460$$

Therefore, the chance of selected the correct 6 numbers is $\frac{1}{18009460}$

➤b.

Let p = $\frac{1}{18009460}$, chance of winning.

$q = 1 - p = 1 - \frac{1}{18009460} = \frac{18009459}{18009460}$, chance of losing.

$E(X) = \$10{,}000{,}000p - \$1q = \frac{1{,}000{,}000}{18009460} - \frac{18009459}{18009460} \approx -\$0.94$

➤c.

w: The prize

$E(X) = wp - \$1q = \frac{w}{18009460} - \frac{18009459}{18009460} = 0.$

Therefore, w = 18,009,459

14.

➤a.

Let X be the random variable equal to the number of possible clubs in her hand.

$\#\mathbf{S} = \binom{52}{7} = 133{,}784{,}560$, total possible number of hands.

$$P\{X = 0\} = \frac{\binom{39}{7}}{\binom{52}{7}} = \frac{15{,}380{,}937}{133{,}784{,}560}$$

$$P\{X=1\} = \frac{\binom{13}{1}\binom{39}{6}}{\binom{52}{7}} = \frac{42{,}414{,}099}{133{,}784{,}560}$$

$$P\{X=2\} = \frac{\binom{13}{2}\binom{39}{5}}{\binom{52}{7}} = \frac{44{,}909{,}046}{133{,}784{,}560}$$

$$P\{X=3\} = \frac{\binom{13}{3}\binom{39}{4}}{\binom{52}{7}} = \frac{23{,}523{,}786}{133{,}784{,}560}$$

$$P\{X=4\} = \frac{\binom{13}{4}\binom{39}{3}}{\binom{52}{7}} = \frac{6{,}534{,}385}{133{,}784{,}560}$$

$$P\{X=5\} = \frac{\binom{13}{5}\binom{39}{2}}{\binom{52}{7}} = \frac{953{,}667}{133{,}784{,}560}$$

$$P\{X=6\} = \frac{\binom{13}{6}\binom{39}{1}}{\binom{52}{7}} = \frac{66{,}924}{133{,}784{,}560}$$

$$P\{X=7\} = \frac{\binom{13}{7}\binom{39}{0}}{\binom{52}{7}} = \frac{1{,}716}{133{,}784{,}560}$$

$$E(X) = (0)\frac{15,380,937}{133,784,560} + (1)\frac{42,414,099}{133,784,560} + (2)\frac{44,909,046}{133,784,560} + (3)\frac{23,523,786}{133,784,560} +$$

$$(4)\frac{6,534,385}{133,784,560} + (5)\frac{953,667}{133,784,560} + (6)\frac{66,924}{133,784,560} + (7)\frac{1,716}{133,784,560} = 1.75$$

➤b.

E: The event that the hand contains 3 kings and 2 queens or 3 diamonds.

A: The event that the hand contains 3 kings and 2 queens.

B: The event that the hand contains 3 diamonds.

E = **A**∪**B**

P(**E**) = P(**A**∪**B**) = P(**A**) + P(**B**) - P(**A**∩**B**)

$$P(\mathbf{A}) = \frac{\binom{4}{3}\binom{4}{2}\binom{44}{2}}{\binom{52}{7}} = \frac{22704}{\binom{52}{7}}$$

$$P(\mathbf{B}) = \frac{\binom{13}{3}\binom{39}{4}}{\binom{52}{7}} = \frac{23,523,786}{\binom{52}{7}}$$

A∩**B**: The event that 3 kings, 2 queens and 3 diamonds are in the hand.

This can happen in 3 different ways:

Case 1:

1 king of diamonds and 2 other kings , 1 queen of diamonds and 1 other queen,
1 diamond that is not a king nor queen , and 1 other non king, queen nor a diamond.

The number of ways this can happen is

$$\binom{3}{2}\binom{3}{1}\binom{11}{1}\binom{33}{1} = (3)(3)(11)(33) = 3,267.$$

or

Case 2:

1 king of diamonds and 2 other kings , 2 queens that are not diamonds , 2 diamonds that are not king nor queen.

The number of ways this can happen is

$$\binom{3}{2}\binom{3}{2}\binom{11}{2} = (3)(3)(55) = 495.$$

or

Case 3:

3 kings that are not diamonds, 1 queens that is a diamond and the other queen is not a diamond, 2 diamonds that are not king nor queen.

The number of ways this can happen is

$$\binom{3}{3}\binom{3}{1}\binom{11}{2} = (1)(3)(55) = 165.$$

$\mathbf{A}\cap\mathbf{B}$ = 3267 + 495 + 165 = 3927

$$P(\mathbf{A}\cap\mathbf{B}) = \frac{3926}{\binom{52}{7}}$$

$$P(\mathbf{E}) = P(\mathbf{A}) + P(\mathbf{B}) - P(\mathbf{A}\cap\mathbf{B}) = \frac{22704}{\binom{52}{7}} + \frac{23{,}523{,}786}{\binom{52}{7}} - \frac{3926}{\binom{52}{7}} = \frac{23{,}542{,}564}{133{,}784560}$$

15.

Since there are 10 items of one can sell any number of them the number of possible sandwich combinations is $2^{10} = 1024$.

16.

➤a.

Step 1: The number of ways N people birthdays can happen is $\mathbf{S} = 365^N$

Step 2: The number of ways of selecting 2 people out of N is $\binom{N}{2}$.

The total number of ways the 2 people birthday can occur is $365\binom{N}{2}$.

Step 3: The number of ways the remaining N -2 people birthdays can fall on different days is

364(364-1)(364-2)...[364 - (N-2) + 1] = 364(363)...(367 - N)

Step 4: The total number of ways of only 2 people being born on the same day is

$$\binom{N}{2} 365(364)(363)...(367 - N) = \binom{N}{2} {}_{365}P_{N-1}$$

The probability is $({}_{365}P_{N-1})\ 365^{-N}$

➤b

Step 1: The number of ways N people birthdays can happen is **S** = 365^N

Step 2: The number of ways of selecting k people out of N is $\binom{N}{k}$.

The total number of ways the k people birthday can occur is $365\binom{N}{k}$.

Step 3: The number of ways the remaining N -k people birthdays can fall on different days is

364(364-1)(364-2)...[364 - (N-k) + 1] = 364(363)...(365 + k - N)

Step 4: The total number of ways of only k people being born on the same day is

$$\binom{N}{k} 365(364)(363)...(365 + k - N) = \binom{N}{k} {}_{365}P_{N-k+1}$$

The probability is

$$[\binom{N}{k} {}_{365}P_{N-k+1}]\ 365^{-N}.$$

17.

➤a.

Since there are only 4 suits, the number of royal flushes is 4.

➤b.
The number sequences are

(A,2,3,4,5), (2,3,4,5,6), (3,4,5,6,7), (4,5,6,7,8),(5,6,7,8,9),(6,7,8,9,10),
(7,8,9,10,J), (8,9,10,J,Q), (9,10,J,Q,K), (10,J,Q,K,A) .

Since the are 4 suits for each selection the total number of straight flushes is

4(10) = 40.

➤c.
There are 13 cards in each suit. Therefore, the are 13 four card of equal rank. For each possible For each of these, there are 12(4) possibilities for the fifth card.

There fore the total number possible is

13(12)(4) = 624.

➤d.
Step 1: The number of ways three cards of equal rank can be selected is

$$\binom{4}{3} = 4.$$

Since there are 13 cards per suit, the number of three cards of equal rank is

13(4) = 52.

Step 2: The number of ways two cards of equal rank can be selected is

$$\binom{4}{2} = 6$$

Since there are 12 remaining cards per suit, the number of two cards of equal rank is

12(6) = 72.

Therefore, the total number of full houses is

52(72) = 3,744.

➤e.
There are 13 cards in each suit. The number of ways of selecting 5 cards out of 13 cards is

$$\binom{13}{5} = 1,287.$$

There are 4 suits. Therefore, the total number of flushes is 4(1,287) = 5,148.

However, we must subtract straights. Since there are 40 straights, we have

5,148 - 40 = 5,108.

➤f.
All five card hands in sequence can be represented as

(A,2,3,4,5), (2,3,4,5,6), (3,4,5,6,7), (4,5,6,7,8), (5,6,7,8,9), (6,7,8,9,10), (7,8,9,10,J),

(8,9,10,J,Q), (9,10,J,Q,K), (10,J,Q,K,A).

Now each of these can occur in 4^5 = 1,024 ways. Therefore, the total number of possible straight hands is

10(1,024) = 10,240.

However, this number includes straight flushes. There are 40 such hands. Since we want only mixed, the number of straights is

10,240 - 40 = 10,200.

➤g.
Assume the hand is in the following order: first 2 cards are a pair of equal rank, the second 2 cards are a different pair of equal rank, and the fifth card is different than the pairs.

Step 1: The number of ways the first pair can happen is

$$13\binom{4}{2} = 13(6) = 78.$$

Step 2: The number of ways the second pair can happen (different than the first pair) is

$$12\binom{4}{2} = 12(6) = 72.$$

Step 3: The number of ways the fifth card can occur different than the 2 pairs is

11(4) = 44.

Step 4: The number of ways this five card hand can occur is

78(72)(44)/2 =123,552.

We divided by 2 because our counting method allows for each pair to occur twice (A,A), (5,5) and (5,5),(A,A).

➤h.
Step 1: Assume the first two cards is a pair of equal rank. The number of ways this can happen is

$13\binom{4}{2} = 13(6) = 78.$

Step 2: Assume the remaining three cards are all different. The number of ways this can happen is

(4)(12)(4)(11)(4)(10) = 84,480.

Here order is important. We need to factor out each permutation.

To do this we divide by 3! = 6:

84,480/6 = 14,080.

Step 3: Therefore, the total number of different hands consisting of 1 pair is

14,080(78) = 1,098,240.

18.
➤a.
The total number of possible hands for the first hand dealt is

$\binom{52}{5}.$

The total number of possible hands for the second hand dealt is

$\binom{47}{5}.$

There, the total number of possible pair of hands is

$\binom{52}{5}\binom{47}{5}$.

➤b.
Step 1:

K_1: The event the first hand dealt contains exactly 1 king.

K_2: The event the second hand dealt contains exactly 1 king.

$K_1 \cap K_2$: The event both hands contain exactly 1 king each.

$\#\mathbf{K}_1 = 4\binom{48}{4} = 4(194580) = 778{,}320$, different hands containing 1 king only.

Step 2: $P(\mathbf{K}_1 \cap \mathbf{K}_2) = P(\mathbf{K}_1)P(\mathbf{K}_2|\mathbf{K}_1)$

$P(\mathbf{K}_1) = 4\binom{48}{4}/\binom{52}{5} \approx 0.3$

$P(\mathbf{K}_2|\mathbf{K}_1) = 3\binom{44}{4}/\binom{47}{5} \approx 0.27$

$P(\mathbf{K}_1 \cap \mathbf{K}_2) = P(\mathbf{K}_1)P(\mathbf{K}_2|\mathbf{K}_1) \approx 0.3(0.27) = 0.081$

➤c.
$K_1 \cup K_2$: the event that at least 1 hand has exactly 1 king.

$P(\mathbf{K}_1 \cup \mathbf{K}_2) = P(\mathbf{K}_1) + P(\mathbf{K}_2) - P(\mathbf{K}_1 \cap \mathbf{K}_2) \approx 0.3 + 0.3 - 0.08 = 0.52.$

19.
Step 1:
K_1: The event the first hand dealt contains exactly 1 king.
K_2: The event the second hand dealt contains exactly 1 king.
K_3: The event the third hand dealt contains exactly 1 king.

$\mathbf{K}_1 \cup \mathbf{K}_2 \cup \mathbf{K}_3$: The event that at least 1 hand contains exactly 1 king.

From Lesson 10, Supplementary problem 13 a, $P(\mathbf{K}_1)$

$P(\mathbf{K}_1 \cup \mathbf{K}_2 \cup \mathbf{K}_3) = P(\mathbf{K}_1) + P(\mathbf{K}_2) + P(\mathbf{K}_3) - P(\mathbf{K}_1 \cap \mathbf{K}_2) - P(\mathbf{K}_1 \cap \mathbf{K}_3) - P(\mathbf{K}_2 \cap \mathbf{K}_3) + P(\mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_3)$

From Problem 18 above,

$P(\mathbf{K}_1 \cup \mathbf{K}_2 \cup \mathbf{K}_3) = P(\mathbf{K}_1) + P(\mathbf{K}_2) + P(\mathbf{K}_3) - P(\mathbf{K}_1 \cap \mathbf{K}_2) - P(\mathbf{K}_1 \cap \mathbf{K}_3) - P(\mathbf{K}_2 \cap \mathbf{K}_3) + P(\mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_3)$

$\approx 0.3 + 0.3 + 0.3 - 0.08 - 0.08 - 0.08 + P(\mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_3)$

$$P(\mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_3) \; \frac{4\binom{48}{4}3\binom{44}{4}2\binom{40}{4}}{\binom{52}{5}\binom{47}{5}\binom{42}{5}} \approx 0.02$$

$P(\mathbf{K}_1 \cup \mathbf{K}_2 \cup \mathbf{K}_3) \approx 0.9 - 0.24 + 0.02 = 0.68$

20.
The following table shows all possible ways the three urns will contain at least 1 ball.

Urn 1 # balls	**Urn 2 # balls**	**Urn 3 # balls**	**Number of ways**
1	1	3	$\binom{5}{1}\binom{4}{1}\binom{3}{3} = 20$
1	2	2	$\binom{5}{1}\binom{4}{2}\binom{2}{2} = 30$
1	3	1	$\binom{5}{1}\binom{4}{3}\binom{1}{1} = 20$
2	1	2	$\binom{5}{2}\binom{3}{1}\binom{2}{2} = 30$
2	2	1	$\binom{5}{2}\binom{3}{2}\binom{1}{1} = 30$
3	1	1	$\binom{5}{3}\binom{2}{1}\binom{1}{1} = 20$
			Total 150

Since there are 3 urns and 5 balls, $\# \mathbf{S} = (3)(3)(3)(3)(3) = 3^5 = 243$.

Therefore, the probability that no urn is empty is 150/243.

21.
Let X be the number of tosses.

X = 3

Urn 1 # balls	**Urn 2 # balls**	**Urn 3 # balls**	**Number of ways**
1	1	1	(3)(2)(1) = 6

X = 4

Urn 1 # balls	**Urn 2 # balls**	**Urn 3 # balls**	**Number of ways**
2	1	1	$\binom{4}{2}\binom{2}{1}\binom{1}{1} = 12$
1	2	1	$\binom{4}{1}\binom{3}{2}\binom{1}{1} = 12$
1	1	2	$\binom{4}{1}\binom{3}{1}\binom{2}{2} = 12$

X = x	**P{X = x}**	**xP{X = x}**
3	$6/3^3 = 6/27$	3(6/27) = 18/27
4	$36/3^4 = 36/81$	4(36/81) = 144/81
5	1 - 6/27 - 36/81 = 27/81	5(27/81) = 135/81
	Total 1	E(X) = 333/81 ≈ 4.11

22.
Assume the first 2 cards are kings. We have the following possibilities:

$\mathbf{E}_1 = (\mathbf{K}_1 \cap \mathbf{D}_1) \cap (\mathbf{K}_2 \cap \mathbf{D}_2') \cap (\mathbf{K}_3' \cap \mathbf{D}_3) \cap (\mathbf{K}_4' \cap \mathbf{D}_4')$

$P(\mathbf{E}_1) = (1/52)(3/51)(12/50)(36/49) = (1296)/(6497400)$

$\mathbf{E}_2 = (\mathbf{K}_1 \cap \mathbf{D}_1) \cap (\mathbf{K}_2 \cap \mathbf{D}_2') \cap (\mathbf{K}_3' \cap \mathbf{D}_3') \cap (\mathbf{K}_4' \cap \mathbf{D}_4)$

$P(\mathbf{E}_2) = (1/52)(3/51)(36/50)(12/49) = (1296)/(6497400)$

$\mathbf{E}_3 = (\mathbf{K}_1 \cap \mathbf{D}_1') \cap (\mathbf{K}_2 \cap \mathbf{D}_2) \cap (\mathbf{K}_3' \cap \mathbf{D}_3) \cap (\mathbf{K}_4' \cap \mathbf{D}_4')$

$P(\mathbf{E}_2) = (3/52)(1/51)(12/50)(36/49) = (1296)/(6497400)$

$\mathbf{E}_4 = (\mathbf{K}_1 \cap \mathbf{D}_1{}')\cap(\mathbf{K}_2 \cap \mathbf{D}_2)\cap(\mathbf{K}_3{}' \cap \mathbf{D}_3{}')\cap(\mathbf{K}_4{}' \cap \mathbf{D}_4)$

$P(\mathbf{E}_2) = (3/52)(1/51)(36/50)(12/49) = (1296)/(6497400)$

$\mathbf{E}_5 = (\mathbf{K}_1 \cap \mathbf{D}_1{}')\cap(\mathbf{K}_2 \cap \mathbf{D}_2{}')\cap(\mathbf{K}_3{}' \cap \mathbf{D}_3)\cap(\mathbf{K}_4{}' \cap \mathbf{D}_4)$

$P(\mathbf{E}_5) = (3/52)(2/51)(12/50)(11/49) = (792)/(6497400)$

The number of ways of selecting 2 kings for 4 positions in the hand is

$$\binom{4}{2} = 6.$$

Therefore, the probability of the event of 2 kings and 2 diamonds where a king can be a diamond

is

$6\{4[(1296)]/[6497400] + (792/6497400)\} = (35856)/6{,}497{,}400$

23.
➤a.
We use

$e \approx 2.718$
$\pi \approx 3.142$

$10! = (10)(9)(8)(7)(6)(5)(4)(3)(2)(1) = 3{,}628{,}800$

For n = 10, $(2\pi)^{1/2}\ 10^{10 + 1/2} e^{-10} \approx [(2)(3.14)(10)]^{1/2}(453{,}999.3) = (62.8)^{1/2}(453{,}999.3) \approx 3{,}597{,}783$

$|n! - (2\pi)^{1/2}\ n^{n + 1/2} e^{-n}|/n!$ $(3{,}628{,}800 - 3{,}597{,}783)/3{,}628{,}800 = 0.00855$ or 0.8% error

➤b.

$$\binom{2n}{n} = \frac{(2n)!}{n!n!}$$

$(2n)! \approx (2\pi)^{1/2}\ (2n)^{2n + 1/2} e^{-2n}$

$n!n! = [(2\pi)^{1/2}\ n^{n + 1/2} e^{-n}][(2\pi)^{1/2}\ n^{n + 1/2} e^{-n}] = 2\pi n^{\ 2n + 1} e^{-2n}$

$$\binom{2n}{n} = \frac{(2n)!}{n!n!} = [(2\pi)^{1/2}\,(2n)^{2n+1/2}\,e^{-2n}]/[2\pi n^{\,2n+1}\,e^{-2n}] = 2^{2n+1}/\,[(2\pi n)^{1/2}]$$

➤c.
Use the above formula for n = 10, we get

$$\binom{20}{10} \approx 2^{21}/\,[(20\pi)^{1/2}] \approx 184{,}756$$

24.
We have the distribution P(X = k) = for k = 0,1...,r :

$$P(X = k) = \frac{\binom{n}{k}\binom{N-n}{r-k}}{\binom{N}{r}}.$$

We have P(X = 0) + P(X = 1) + ... + P(X = r) = 1

$$\frac{\binom{n}{0}\binom{N-n}{r-0}}{\binom{N}{r}} + \frac{\binom{n}{1}\binom{N-n}{r-1}}{\binom{N}{r}} + \dots + \frac{\binom{n}{r}\binom{N-n}{r-r}}{\binom{N}{r}} = 1$$

Therefore,

$$\binom{n}{0}\binom{N-n}{r-0} + \binom{n}{1}\binom{N-n}{r-1} + \dots + \binom{n}{k}\binom{N-n}{r-k} + \dots + \binom{n}{r}\binom{N-n}{0} = \binom{N}{r}$$

25.
➤a.
E: The event that 2 red marbles are selected.

A: The event urn A is selected.
B: The event urn B is selected.

E = (**A**∩**E**)∪(**B** ∩**E**)

P(**E**) = P(**A**∩**E**) + P(**B** ∩**E**) = P(**A**)P(**E**|**A**) + P(**B**)P(**E**|**B**) = (1/2)P(**E**|**A**) + (1/2)P(**E**|**B**)

$$P(\mathbf{E}|\mathbf{A}) = \frac{\binom{n}{k}\binom{N-n}{r-k}}{\binom{N}{r}} = \frac{\binom{5}{2}\binom{15-5}{5-2}}{\binom{15}{5}} = \frac{\binom{5}{2}\binom{10}{3}}{\binom{15}{5}} = (10)(120)/3003 = 1200/3003$$

$$P(\mathbf{E}|\mathbf{B}) \quad \frac{\binom{n}{k}\binom{N-n}{r-k}}{\binom{N}{r}} = \frac{\binom{5}{2}\binom{20-5}{5-2}}{\binom{20}{5}} = \frac{\binom{5}{2}\binom{15}{3}}{\binom{20}{5}} = (10)(455)/15504 = 4550/15504$$

P(**E**) = (1/2)(1200/3003) + (1/2)(4550/15504) ≈0.347

➤b.
P(**A**|**E**) = P(**E**|**A**)P(**A**)/P(**E**) ≈ (1200/3003)(0.5)/(0.347) ≈ 0.58

26.
➤a.
From the hypergeometric distribution:

Number of ways of selecting r_1 from N: $\binom{N}{r_1} = \frac{N!}{(r_1!)(N-r_1)!}$

Number of ways of selecting r_2 from N - r_1 : $\binom{N-r_1}{r_2} = \frac{(N-r_1)!}{(r_2!)(N-r_1-r_2)!}$

Number of ways of selecting r_3 from N - r_1 - r_2 : $\binom{N-r_1-r_2}{r_3} = \frac{(N-r_1-r_2)!}{(r_3!)(N-r_1-r_2-r_3)!}$

::

Number of ways of selecting r_n from N-r_1 - r_2 - r_3 - ... - r_{n-1}:

$$\binom{N-r_1-r_2 \ldots -r_{n-1}}{r_n} =$$

$$\frac{(N - r_1 - r_2 - \ldots - r_{n-1})!}{(r_n!)(N - r_1 - r_2 - r_3 - \ldots - r_n)!} = \frac{(N - r_1 - r_2 - \ldots r_{n-1})!}{(r_n!)(0)!} = \frac{(N - r_1 - r_2 - \ldots r_{n-1})!}{r_n!}$$

The number of ways equals the product of each of the above which gives :

$$\frac{N!}{r_1!r_2!\ldots r_n!}.$$

➤b.
10! = 3,628,800

4! = 24, 3! = 6, 2! = 2, 1! = 1

$$\frac{10!}{4!3!2!1!} = 12{,}600$$

27.
➤a.
Each ball selected can be placed in one of N urns. Since each urn has equal chance of being selected, the total number of possible this can happen is

$(N)(N)\ldots(N) = N^r$

➤b.
{1.2}{}{}
{}{1,2}{}
{}{}{1,2}
{1}{2}{}
{1}{}{2}
{}{1}{2}
{2}{1}{}
{2}{}{1}
{}{2}{1}

28.

$$P(X = x;Y = y;Z = z) = \frac{\binom{n}{x}\binom{m}{y}\binom{q}{z}\binom{N - n - m - q}{r - x - y - z}}{\binom{N}{r}}.$$

29.

N = 52.

$0 \le x \le 2$

$0 \le y \le 2$

$0 \le z \le 2$

$$P(X = x;Y = y;Z = z) = \frac{\binom{4}{x}\binom{4}{y}\binom{4}{z}\binom{40}{2 - x - y - z}}{\binom{52}{2}}.$$

Case 1: No face card is drawn: $P(X =0;Y = 0;Z = 0) = \dfrac{\binom{4}{0}\binom{4}{0}\binom{4}{0}\binom{40}{2}}{\binom{52}{2}} = 780/1326$

Case 2: One face card is drawn:

$P(X=1;Y= 0;Z = 0) = P(X = 0;Y = 1;Z = 0)=P(X=0;Y=0;Z=1) = \dfrac{\binom{4}{1}\binom{4}{0}\binom{4}{0}\binom{40}{1}}{\binom{52}{2}} =$

160/1326

Case 3: Two face cards are drawn:

$P(X = 2;Y = 0;Z = 0) = P(X = 0;Y = 2;Z = 0) = P(X = 0;Y = 0;Z = 2)$ $\dfrac{\binom{4}{2}\binom{4}{0}\binom{4}{0}\binom{40}{0}}{\binom{52}{2}} =$

6/1326

$P(X = 1;Y = 1;Z = 0) = P(X = 1;Y = 0;Z = 1) = P(X = 0;Y = 1;Z = 1) =$

$$\frac{\binom{4}{1}\binom{4}{1}\binom{4}{0}\binom{40}{0}}{\binom{52}{2}} = 16/1326$$

30.

For ball number 1 we have N urns to select from ;
for ball number 2 we have N - 1 urns to select from;
for ball number k we have N - k + 1, where k = 1, ...,N.
To get all possible arrangements we multiply these number:

$N(N - 1)(N - 2)...(1) = N!$

31.

➤a.

The way to show this is the same as shown in problem 26 a.

➤b.

Urn 1	**Urn 2**	**Urn 3**	**Urn 4**	**Urn 5**	**Urn 6**
1	2,3		4		
1	2,4		3		
1	3,4		2		
2	1,3		4		
2	1,4		3		
2	3,4		1		
3	1,2		4		
3	1,4		2		
3	2,4		1		
4	1,2		3		
4	1,3		2		
4	2,3		1		

There are 12 ways his can be done.

32.

➤a.

S = {[(a,b,c), ()], [(),(a,b,c)],[(a,b),(c)],[(a,c),(b)],[(b,c),(a)],[(a),(b,c)],[(b),(a,c)],[(c),(a,b)]}

➤b.

The number of ways of selecting m cells from n cells where only 1 particle is placed :

$$\binom{n}{m}\ .$$

Since the order is important for each m cells selected, we have m! possible arrangements.

There fore the total number of possible arrangements with all possible m cells selected is

$$m!\binom{n}{m} = m!\frac{n!}{m!(n-m)!} = \frac{n!}{(n-m)!} = {}_nP_m$$

Since #**S** = n^{-m}, we have P(**E**) = ${}_nP_m n^{-m}$.

➤c.

E = {(a)(b)(), (b)(a)(), ()(a)(b), ()(b)(a), (a)()(b), (b)()(a)}

33.

➤a.

S = {[(a,a,a), ()], [(),(a,a,a)],[(a,a),(a)],[(a),(aa)]}

➤b.

The number of ways of selecting m cells from n cells where we place 1 particle each in the selected cells is

$$\binom{n}{m}.$$

Therefore,

$$\frac{\binom{n}{m}}{\binom{n+m-1}{m}}.$$

➤c.
E = {(a)(a)()(), (a)()(a)(),(a)()()(a),()(a)(a)(),()(a)()(a),()()(a)(a)}

34.
➤a.
S = {[(a),(a),(a),(),()], [(a),(a),(),(a),()], [(a),(a),(),(),(a)],[(a),(),(a),(a),()],[(a)()(a)()(a)], [(a),(),(),(a),(a)], [(),(a),(a),(a),()], [()(a)(a)()(a)], [()(a)()(a)(a)], [()()(a)(a)(a)]}

➤b.
E = {[(a)(a)...(a)...()()()()], ...}.

Since we have a total of n cells, the first contiguous leaves an additional (n - m) contiguous cells. Therefore,

#**E** = n - m + 1.

35.
➤a.
m = 4, number is distinguishable kings.

n = 6, number of players.

Step 1: The total number of ways the kings can be distributed to the 6 players is 6^4.

Step 2: The total number of ways the kings can be distributed evenly to the players is ${}_6P_4 = 6!/2!$.

Step 3: The probability that the kings can be distributed evenly to the players is

${}_6P_4 / 6^4 = (6!/2!)/6^4 = (6)(5)(4)(3)/[(6)(6)(6)(6)] = 10/36 = 5/18.$

➤b.
m = 4, number is indistinguishable kings.

n = 6, number of players.

Step 1: The total number of ways the kings can be distributed to the 6 players is

$$\binom{n+m-1}{m} = \binom{6+4-1}{4} = 126$$

Step 2: The total number of ways the indistinguishable kings can be distributed evenly to the players is

$$\binom{n}{m} = \binom{6}{4} = 15.$$

Step 3: The probability that the kings can be distributed evenly to the players is

15/126 = 5/42

36.

➤a.

X = 1,2,3,4.

(X = 1), the event that 1 player was dealt all the kings.
(X = 2), the event that 2 players were dealt all the kings.
(X = 3), the event that 3 players were dealt all the kings.
(X = 4), the event that 4 players were dealt all the kings.

n = 6
m = 4
#**S** = 6^4

X = 1: Since there are 6 players this can happen 6 different ways. Therefore,

$$P(X = 1) = (6)\frac{4!}{4!0!0!0!0!0!}6^{-4} = 6/6^{-4}$$

X = 2:

Step 1: Start with 2 selected players. The table shows how 4 kings are distributed between them:

Player 1	**Player 2**
1	3
2	2
3	1

Step 2: For these 2 players, the number of ways 4 distinguishable kings can be distributed to the 2 players is

$$\frac{4!}{1!3!0!0!0!0!} + \frac{4!}{2!2!0!0!0!0!} + \frac{4!}{3!1!0!0!0!0!} = 4 + 6 + 4 = 14.$$

Step 3: The total numbers of selecting 2 players out of 6 is

$$\binom{6}{2} = 15.$$

Therefore, $P(X = 2) = (15)(14)/6^4 = 210/6^4$

X = 3:

Step 1: Start with 3 selected players. The table shows how 4 kings are distributed between them:

Player 1	**Player 2**	**Player 3**
1	1	2
1	2	1
2	1	1

The number of ways the 4 kings can be distributed to the 3 selected players is

$$\frac{4!}{1!1!2!0!0!0!} + \frac{4!}{1!2!1!0!0!0!} + \frac{4!}{2!1!1!0!0!0!} = 12 + 12 + 12 = 36.$$

Step 2: The number of ways of selecting 3 players out of 6 is

$$\binom{6}{3} = 20.$$

Therefore, $P(X = 3) = (20)(36)/6^4 = 720/6^4$

X = 4:
Step 1: Start with 4 selected players. There is only

$$\frac{4!}{1!1!1!1!} = 24$$

ways 4 players can be dealt 4 kings.

Step 2: The total number of ways 4 players can be selected from 6 is

$$\binom{6}{4} = 15.$$

Therefore, $P(X = 4) = (15)(24)/6^4 = 360/6^4$

➤b.
Y = 1,2,3,4.

(Y = 1), the event that 1 player was dealt all the kings.
(Y = 2), the event that 2 players were dealt all the kings.
(Y = 3), the event that 3 players were dealt all the kings.
(Y = 4), the event that 4 players were dealt all the kings.
n = 6
m = 4

$$\#\mathbf{S} = \binom{n + m - 1}{m} = \binom{6 + 4 - 1}{4} = 126$$

Y = 1:
There is only 1 way to deal all 4 indistinguishable kings to 1 player. Since there are 6 players:

P(Y = 1) = 6/126.

Y = 2:
Step 1: First select 2 players. The following shows the number of ways of distributing the 4 kings among the 2 players:

[(k,k,k),(k)],[(k,k),(k,k)],[(k),(k,k,k)]

Step 2: The number of ways of selecting 2 players from 6 players is

$$\binom{6}{2} = 15.$$

Therefore, P(Y = 2) = (15)(3)/126 = 45/126

Y = 3:
Step 1: First select 3 players. The following shows the number of ways of distributing the 4 kings among the 3 players:

[(k),(k),(k,k)], [(k),(k,k),(k)], [(k,k),(k),(k)].

Step 2: The number of ways of selecting 3 players out of 6 is

$$\binom{6}{3} = 20.$$

P(Y = 3) = (20)(3)/126 = 60/126

Y = 4:
Step 1: First select 4 players. The following shows the number of ways of distributing the 4 kings among the 4 players:

[(k),(k),(k),(k)].

Step 2: The total number of ways 4 players can be selected from 6 is

$$\binom{6}{4} = 15.$$

Therefore, P(Y = 4) = 15(1)/126 = 15/126

37.
➤a.
This problem can be solved using the Maxwell- Bolzmann statistics where

n = 364, the number of days (cells) in the year.
m: the number of people (particles) randomly selected.

$\#\mathbf{S} = 364^m$

E: At least 2 people are born on the same day.

E′: Each person randomly selected is born on a different day.

#**E**′: the number of ways of selecting m different days from the 364 days. Since order of the days is important,

$\#\mathbf{E}' = {}_{364}P_m$

$P(\mathbf{E}') = {}_{364}P_m/(364^m)$

Therefore, $P(\mathbf{E}) = 1 - P(\mathbf{E}') = 1 - ({}_{364}P_m)/(364^m)$

➤b.

$\mathbf{E}_3$: The event that at least 3 people were born on the same day.

$\mathbf{E}_o$: The event that no one were born on the same day.

$\mathbf{E}_2$: The event that exactly 2 people were born on the same day.

Step 1: $\mathbf{E}_3'$: The event that less than 3 people were born on the same day

$\mathbf{E}_3' = \mathbf{E}_0 \cup \mathbf{E}_2$

$P(\mathbf{E}_3') = P(\mathbf{E}_0) + P(\mathbf{E}_2)$

From a. above,

$P(\mathbf{E}_0) = ({}_{364}P_m)/(364^m)$

Step 2: We will compute $\#\mathbf{E}_2$.

Select a fixed day. The number ways 2 people can be born on the same day is

$$\binom{m}{2}$$

The number of ways the remaining people have different birthday is ${}_{363}P_{m-2}$.

The number of days in the year is 364.

Therefore,

$$\#\mathbf{E}_2 = (364)\binom{m}{2}{}_{363}P_{m-2}$$

$$P(\mathbf{E}_2) = [(364)\binom{m}{2}{}_{363}P_{m-2}]/(364^m) =$$

$$P(\mathbf{E}_3') = P(\mathbf{E}_0) + P(\mathbf{E}_2) = ({}_{364}P_m)/(364^m) + [(364)\binom{m}{2}{}_{363}P_{m-2}]/(364^m) =$$

$$\{({}_{364}P_m) + [(364)\binom{m}{2}{}_{363}P_{m-2}]\}/364^m$$

$$P(\mathbf{E}_3) = 1 - P(\mathbf{E}_3) = 1 - \{({}_{364}P_m) + [(364)\binom{m}{2}{}_{363}P_{m-2}]\}/364^m$$

➤c.

$\mathbf{E}_r$: The event that exactly r people are born on the same day.

Select a fixed day. The number ways r people can be born on the same day is

$$\binom{m}{r}$$

The number of ways the remaining m - r people have different birthday is

${}_{363}P_{m\ -\ r}$

The number of days in the year is 364.

Therefore,

$$\#\mathbf{E}_r = (364)\binom{m}{r}{}_{363}P_{m\ -\ r}$$

$$P(\mathbf{E}_r) = [(364)\binom{m}{r}{}_{363}P_{m\ -\ r}]/(364^m)$$

➤d.

From a., we have $P(\mathbf{E}) = 1 - ({}_{364}P_m)/(364^m)$.

$1 - ({}_{364}P_m)/(364^m \geq 0.50$

$- ({}_{364}P_m)/(364^m \geq -1 + 0.50 = -0.50$

$({}_{364}P_m)/(364^m \leq 0.50$

Since we need to find m for the above, we compute $({}_{364}P_m)/(364^m$ for values of m = 5k, k = 1,2,...

$m = 5: ({}_{364}P_5)/(364^5) \approx 0.97$

$m = 10: ({}_{364}P_{10})/(364^{10}) \approx 0.88$

$m = 15: ({}_{364}P_{15})/(364^{15}) \approx 0.75$

$m = 20: ({}_{364}P_{20})/(364^{20}) \approx 0.58$

Now we use values for m = 22,24,26,...

m = 22: $({}_{364}P_{22})/(364^{22}) \approx 0.52$

m = 24: $({}_{364}P_{24})/(364^{24}) \approx 0.46$

m = 23: $({}_{364}P_{23})/(364^{23}) \approx 0.49$

38.
E: The event that r identical automobiles are repaired on exactly 1 day

Select a fixed day. The number of ways of selected r indistinguishable automobiles that have been repaired on a single day is 1.

We assume the remaining m - r automobiles are repaired no more than once a day. Therefore,

the number of ways these automobiles can be distributed throughout the year is

$$\binom{363}{m-r} .$$

Since we assume there are 364 days in the year,

$$\#\mathbf{E} = 364\binom{363}{m-r} .$$

Since we are using the Bose-Einstein statistical model,

$$P(\mathbf{E}) = 364\binom{363}{m-r} \Big/ \binom{364+m-1}{m} .$$

39.
$\mathbf{R}_k$: The event that the kth ball is red.

Assume the following order occurred:

$\mathbf{R}_1 \cap \mathbf{R}_2 \cap \mathbf{R}_3 \cap \mathbf{R}_4' \cap \mathbf{R}_5' \cap \mathbf{R}_6'$

$P(\mathbf{R}_1 \cap \mathbf{R}_2 \cap \mathbf{R}_3\, \mathbf{R}_4' \cap \mathbf{R}_5' \cap \mathbf{R}_6') =$

$P(\mathbf{R}_1)P(\mathbf{R}_2|\mathbf{R}_1)P(\mathbf{R}_3|\mathbf{R}_1 \cap \mathbf{R}_2)P(\mathbf{R}_4'|\mathbf{R}_1 \cap \mathbf{R}_2 \cap \mathbf{R}_3)P(\mathbf{R}_5'|\mathbf{R}_1 \cap \mathbf{R}_2 \cap \mathbf{R}_3 \cap \mathbf{R}_4')P(\mathbf{R}_6'|\mathbf{R}_1 \cap \mathbf{R}_2 \cap \mathbf{R}_3 \cap \mathbf{R}_4' \cap \mathbf{R}_5')$

$= (6/12)(5/11)(4/10)(6/9)(5/8)(4/7) = 14400/665280 = 5/231$

Now the number of possible arrangements we can select 3 red balls from 6 selections is

$$\binom{6}{3} = 20.$$

Therefore, P(**E**) = 20(5/231) = 100/231

40.

A: The event that only 1 player was dealt 2 kings.

$\mathbf{K}_k$: k kings were dealt. (k = 1,2,3,4)

We want to find $P(\mathbf{K}_4|\mathbf{A})$.

Using Baye's theorem, we have

$P(\mathbf{K}_4|\mathbf{A}) = \mathbf{P(A|K_4)}P(\mathbf{K}_4)/P(\mathbf{A})$

Step 1: We will use the *Maxwell-Boltzmann Statistics* model, where
m = 4
n = 6.

The probability that the 6 players' hands contain r_1,r_2,r_3,r_4,r_5,r_6 kings where

$$r_1 + r_2 + r_3 + r_4 + r_5 + r_6 = 4$$

and

$r_k \geq 0$ (k = 1,.. , 6) is

$$\frac{4!}{r_1!r_2!r_3!r_4!r_5!r_6!}6^{-4}.$$

Step 2: Assume player 1 received 2 kings and players 2 and 3 received 1 king each. The probability of this event happening is

$$\frac{4!}{2!1!1!0!0!0!}6^{-4} = (12)6^{-4} .$$

For player 1 the number ways the remain 5 players can be dealt 2 kings is

$$\binom{5}{2} = 10 .$$

Since there are 6 players, the number of ways 1 player is dealt 2 kings and each of the remaining 2 kings dealt to 2 other players is 6(10) = 60. Therefore,

$\mathbf{P}(\mathbf{A}|\ \mathbf{K}_4) = (60)(12)6^{-4} = 720(6^{-4})$.

Step 2: Since there are 6 players and 5 cards are dealt to each player, 30 cards are dealt without replacement. The number of different ways of selecting all 4 kings when dealing these 30 cards is

$$\binom{4}{4}\binom{48}{26} = \binom{48}{26}$$

The number of different ways of dealing 30 cards is

$$\binom{52}{30}.$$

Therefore,

$$\mathrm{P}(\mathbf{K}_4) = \frac{\binom{48}{26}}{\binom{52}{30}}$$

Step 3: $\mathbf{A} = (\mathbf{A}\cap\mathbf{K}_1)\cup(\mathbf{A}\cap\mathbf{K}_2)\cup(\mathbf{A}\cap\mathbf{K}_3)\cup(\mathbf{A}\cap\mathbf{K}_4) = \phi\cup(\mathbf{A}\cap\mathbf{K}_2)\cup(\mathbf{A}\cap\mathbf{K}_3)\cup(\mathbf{A}\cap\mathbf{K}_4) =$
$(\mathbf{A}\cap\mathbf{K}_2)\cup(\mathbf{A}\cap\mathbf{K}_3)\cup(\mathbf{A}\cap\mathbf{K}_4)$

$\mathrm{P}(\mathbf{A}) = \mathrm{P}[(\mathbf{A}\cap\mathbf{K}_2)\cup(\mathbf{A}\cap\mathbf{K}_3)\cup(\mathbf{A}\cap\mathbf{K}_4)] = \mathrm{P}(\mathbf{A}\cap\mathbf{K}_2) + \mathrm{P}(\mathbf{A}\cap\mathbf{K}_3) + \mathrm{P}(\mathbf{A}\cap\mathbf{K}_4) =$

$\mathrm{P}(\mathbf{A}|\mathbf{K}_2)\mathrm{P}(\mathbf{K}_2) + \mathrm{P}(\mathbf{A}|\mathbf{K}_3)\mathrm{P}(\mathbf{K}_3) + \mathrm{P}(\mathbf{A}|\mathbf{K}_4)\mathrm{P}(\mathbf{K}_4)$

Assume only 2 kings were dealt and both were dealt to 1 player. The probability of this event happening is

$$\mathbf{P}(\mathbf{A}|\ \mathbf{K}_2) = (6)\frac{2!}{2!0!0!0!0!0!}6^{-2} = 6^{-1} ,$$

$$P(\mathbf{K}_2) = \frac{\binom{4}{2}\binom{48}{28}}{\binom{52}{30}} = \frac{6\binom{48}{28}}{\binom{52}{30}} .$$

Assume only 3 kings were dealt and 2 kings to 1 player. Select 1 arrangement. The probability of this happening is

$$\frac{3!}{2!1!0!0!0!0!}6^{-3} = (3)6^{-3}$$

Since there are 6 players the number of such arrangements is 6(5) = 30. Therefore,

$\mathbf{P(A|\ K_3)} = (6)(3)6^{-3} = (3)6^{-2}$,

$$P(\mathbf{K}_3) = \frac{\binom{4}{3}\binom{48}{27}}{\binom{52}{30}} = \frac{4\binom{48}{27}}{\binom{52}{30}} .$$

Therefore,

$P(\mathbf{A}) = P(\mathbf{K}_2)P(\mathbf{A}|\mathbf{K}_2) + P(\mathbf{K}_3)P(\mathbf{A}|\mathbf{K}_3) + P(\mathbf{K}_4)P(\mathbf{A}|\mathbf{K}_4) =$

$$\frac{6\binom{48}{28}}{\binom{52}{30}}6^{-1} + \frac{4\binom{48}{27}}{\binom{52}{30}}(3)6^{-2} + \frac{\binom{48}{26}}{\binom{52}{30}}720(6^{-4}) =$$

$P(\mathbf{K}_4|\mathbf{A}) = \mathbf{P(A|K_4)}P(\mathbf{K}_4)/P(\mathbf{A}) =$

$$\frac{\binom{48}{26}}{\binom{52}{30}}720(6^{-4}) \ \left[\frac{6\binom{48}{28}}{\binom{52}{30}}6^{-1} + \frac{4\binom{48}{27}}{\binom{52}{30}}(3)6^{-2} + \frac{\binom{48}{26}}{\binom{52}{30}}720(6^{-4}) \ \right]^{-1} =$$

$$20\binom{48}{26}(6^{-2}) \ \left[\binom{48}{28} + 2\binom{48}{27}6^{-1} + 20\binom{48}{26}(6^{-2}) \ \right]^{-1}$$

$5\binom{48}{26}$ $\left[\ 9\binom{48}{28} + 3\binom{48}{27} + 5\binom{48}{26}\ \right]^{-1.}$

41.

➤a.

$(\overline{X} \le 1.2) = [\overline{X} = (1 + 1 + ... + 1)/10] \cup [\overline{X} = (2 + 1 + ... + 1)/10] \cup [\overline{X} = (2 + 2 + ... + 1)/10] =$

$[\overline{X} = 10/10] \cup [\overline{X} = 11/10] \cup [\overline{X} = 12/10] = [\overline{X} = 1] \cup [\overline{X} = 1.1] \cup [\overline{X} = 1.2] =$

$P(\overline{X} \le 1.2) = P[\overline{X} = 1] + P[\overline{X} = 1.1] + P[\overline{X} = 1.2] = (1/6)^{10} + \binom{10}{1}(1/6)^{10} + \binom{10}{2}(1/6)^{10} \approx$

$56(1/6)^{10}$

➤b.

$(\overline{X} \le 1.2) = [\overline{X} = (1 + 1 + ... + 1)/100] \cup [\overline{X} = (2 + 1 + ... + 1)/100] \cup [\overline{X} = (2 + 2 + ... + 1)/10] =$

$[\overline{X} = 10/100] \cup [\overline{X} = 11/10] \cup [\overline{X} = 12/10] = [\overline{X} = 1] \cup [\overline{X} = 1.1] \cup [\overline{X} = 1.2] =$

$P(\overline{X} \le 1.2) = P[\overline{X} = 1] + P[\overline{X} = 1.1] + P[\overline{X} = 1.2] = (1/6)^{100} + \binom{100}{1}(1/6)^{100} + \binom{100}{2}(1/6)^{100} =$

$5051(1/6)^{100}$