Statistical Inference Theory
Lesson 37
Decision Theory Using $\overline{P}$

37.1 - Real Life Applications

37.1 - Problem 1:

➤(a).
Step 1:The alternative to H_a, is H_o.

Step 2: The alternative to H_a is $p \geq 0.10$.

Step 3: Therefore, H_o: $p \geq 0.10$

➤(b).
Since the sample proportion $\overline{p} = 0.12 > 0.08$, the decision rule requires that you reject H_a.

➤(c).
Step 1: Since $p = 0.09$, H_a ($p < 0.10$) is true.

Step 2: Since the sample proportion $\overline{p} = 0.12 > 0.08$, the decision rule requires that you reject H_a.

Step 3: Since we are rejecting H_a which is true, a Type II error has occurred.

➤(d).
Step 1: Since $p = 0.11$, H_o ($p \geq 0.10$) is true and H_a is false.

Step 2: Since the sample proportion $\overline{p} = 0.12 > 0.08$, the decision rule requires that you reject H_a.

Step 3: Since we are rejecting H_a which is false, no error has occured.

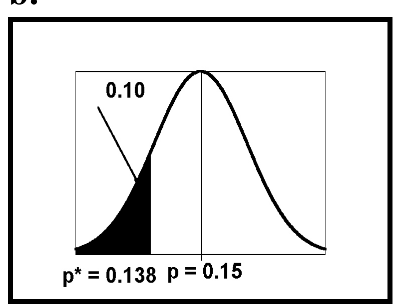
37.1 - Problem 2:

➤a. State H_o and H_a
Since the challenge to the claim $p = 0.15$ is $p < 0.15$, we have

H_o: $p = 0.15$
H_a: $p < 0.15$

➤b.
Since we are interested in a type I error, we assume $p = 0.15$ and need to find p^* so that $P\{\overline{P} < p^*\} = 0.10$.

b.



$$\sigma_{\overline{P}} = \sqrt{\frac{p(1-p)}{N}} = \sqrt{\frac{(0.15)(0.85)}{1500}} \approx 0.01$$

We need the formula $p^* = p + z\sigma_{\overline{p}} = 0.15 + z(0.01)$.

From the normal distribution for the area 0.5 - 0.10 = 0.4 gives
z = -1.28.

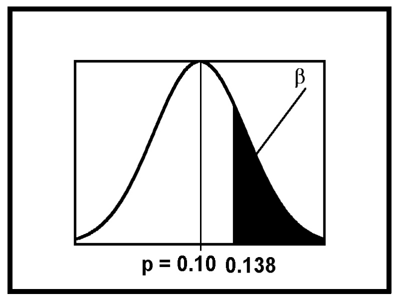
$p^* = 0.15 - 1.28(0.01) \approx 0.14$

The decision rule is restated as:

DR: From the sample, let $\overline{P}$ represent the proportion of tires that lasted more than 35,000 miles. If $0.14 \le \overline{P}$ then accept the claim on the warranty; otherwise reject the claim.

➤c.
A Type II error assumes that H_a is true but rejected.

c.



Step 1: p = 0.10

Step 2: $\sigma_{\overline{P}} = \sqrt{\frac{p(1-p)}{N}} = \sqrt{\frac{(0.1)(0.9)}{1500}} \approx 0.01.$

Step 3: From (b). we use p* = 0.138 for our decision rule.

Step 4: We use the formula $z = \frac{p^* - p}{\sigma_{\overline{p}}} = \frac{0.138 - 0.10}{0.01}. \approx 3.8$

Step 5: From the normal distribution : $P\{\overline{P} \ge 0.138\} = P\{Z \ge 3.8\} = 0.5 - 0.4999 \approx 0 = \beta$

37.1 - Problem 3:

➤a.
Since there is not a one-sided challenge to the claim p = 0.17, we have
H_o: p = 0.17
H_a: p $\neq$ 0.17

➤b.
Since we are interested in a type I error, we assume p = 0.17 and need to find e* so that

$P\{\overline{P} > 0.17 + e^*\} = 0.05/2 = 0.025.$

$$\sigma_{\overline{P}} = \sqrt{\frac{p(1 - p)}{N}} = \sqrt{\frac{(0.17)(0.83)}{500}} \approx 0.02$$

We need the formula $e^* = z\sigma_{\overline{p}} = z(0.02)$.

From the normal distribution for the area 0.5 - 0.025 = 0.475 gives z = 1.96.

$e^* = 1.96(0.02) \approx 0.04$.

0.17 -04 and 0.17 + 0.04 equals 0.13 and 0.21

We restate the decision rule :

If the proportion $\overline{P}$ of the sample that are wearing seat belts is between 0.13 and 0.21 then accept the claim of the broadcaster; otherwise reject the claim.

➤c.
A Type II error assumes that H_a is true but rejected.

Step 1:p = 0.19

Step 2: $\sigma_{\overline{P}} = \sqrt{\frac{p(1 - p)}{N}} = \sqrt{\frac{(0.23)(0.77)}{500}} \approx 0.02$

Step 3:The probability of a Type II error $\beta = P\{0.13 \le \overline{P} \le 0.21\}$ where p = 0.23.

Step 4:We use the formulas:

$$z = \frac{p^* - p}{\sigma_{\overline{P}}} = \frac{0.13 - 0.23}{0.02} = -5$$

$$z = \frac{p^* - p}{\sigma_{\overline{P}}} = \frac{0.21 - 0.23}{0.02} = -1$$

Step 5:From table C: $\beta = P\{0.13 \le \overline{P} \le 0.21\} = P\{-5 \le Z \le -1\} = 0.5 - 0.3413 \approx 0.16$

Supplementary Problems

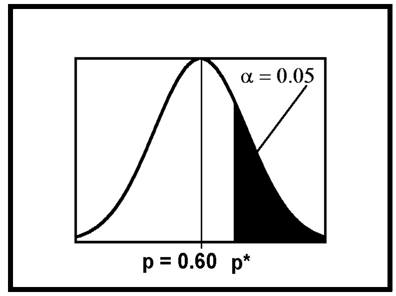
1.

We state the decision rule as: *If $p^* < \overline{P}$ then reject H_o and accept H_a. If $\overline{P} < p^*$ then reject H_a .*

<u>Type I Error</u>

p = 0.60

1.



$$\sigma_{\overline{P}} = \sqrt{\frac{p(1-p)}{N}} = \sqrt{\frac{(0.6)(0.4)}{N}} \approx \frac{0.49}{\sqrt{N}}$$

$$p^* = 0.6 + z\sigma_{\overline{p}} = 0.6 + z\frac{0.49}{\sqrt{N}}$$

From the normal distribution, for the area =05 - 0.05 = 0.45:
z = 1.64.

$$p^* = 0.6 + (1.64)\frac{0.49}{\sqrt{N}} = 0.6 + \frac{0.8}{\sqrt{N}}$$

<u>Type II Error</u>

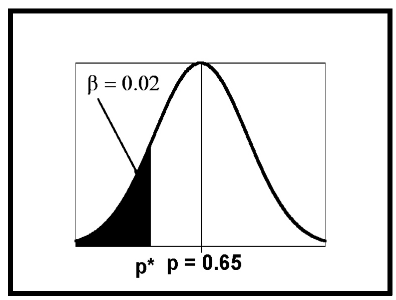
p = 0.65

$$\sigma_{\overline{P}} = \sqrt{\frac{p(1-p)}{N}} = \sqrt{\frac{(0.65)(0.35)}{N}} \approx \frac{0.48}{\sqrt{N}}$$

From the normal distribution, for the area = 0.5 - 0.02 = 0.48: z = -2.06

$$p^* = 0.65 + (-2.06)\frac{0.48}{\sqrt{N}} = 0.65 - \frac{1}{\sqrt{N}}$$

1.



Setting these two equations equal: $0.6 + \frac{0.8}{\sqrt{N}} = 0.65 - \frac{1}{\sqrt{N}}$

$$0.6\sqrt{N} + 0.8 = 0.65\sqrt{N} - 1$$

$$0.05\sqrt{N} = 1.8$$

$$\sqrt{N} = 36$$

$$N = 1296$$

$$p^* = 0.65 - \frac{1}{\sqrt{N}} = 0.65 - \frac{1}{36} \approx 0.62$$

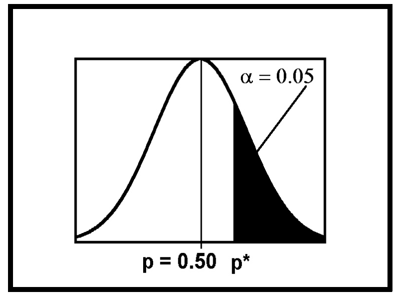
We restate the decision rule as:

Take a sample of size N = 1296. If 0.62 < $\overline{P}$ then reject H_0 and accept H_a. If $\overline{P}$ < 0.62 then reject H_a.

2.
$H_o: p \le 0.5$

H_a: $p > 0.5$

2.



Type I error:

$p = 0.5$

$$\sigma_{\bar{p}} = \sqrt{\frac{p(1-p)}{N}} = \sqrt{\frac{(0.5)(0.5)}{N}} = \frac{0.5}{\sqrt{N}}$$

From the normal distribution, for the area = 0.5 - 0.05 = 0.45:
z = -1.64.

$$p^* = 0.5 + (1.64)\frac{0.5}{\sqrt{N}} = 0.5 + \frac{0.82}{\sqrt{N}}$$

Type II error:

$p = 0.52$

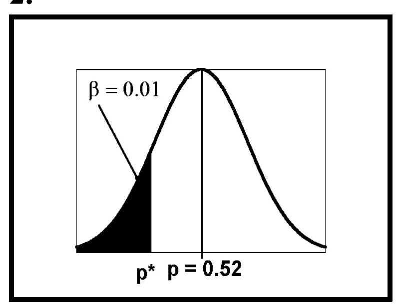
$$\sigma_{\bar{p}} = \sqrt{\frac{p(1-p)}{N}} = \sqrt{\frac{(0.52)(0.48)}{N}} \approx \frac{0.5}{\sqrt{N}}$$

From the normal distribution, for the area = 0.5 - 0.01 = 0.49: z = -2.33

$$p^* = 0.52 + (-2.33)\frac{0.5}{\sqrt{N}} = 0.52 - \frac{1.17}{\sqrt{N}}$$

Setting these two equations equal:

2.



$$0.5 + \frac{0.82}{\sqrt{N}} = 0.52 - \frac{1.17}{\sqrt{N}}$$

$$0.5\sqrt{N} + 0.82 = 0.52\sqrt{N} - 1.17$$

$$0.02\sqrt{N} = 1.99$$

$$\sqrt{N} \approx 100$$

$$N = 10{,}000$$

$$p^* = 0.5 + \frac{0.82}{\sqrt{N}} = 0.5 + \frac{0.82}{100} \approx 0.51$$

We restate the decision rule as:

D.R. Take a sample of size N = 10,000. If at least 51% of the voters say they are in favor of free trade, she will state in her election advertisements that she supports free trade. However, if less than 51% say they are in favor of free trade, she will state in her election advertisements that she does not support free trade.

3.

Type I error:

$p = 0.6$

$$p^* = 0.6 + z\sqrt{\frac{(0.6)(0.4)}{900}} \approx 0.6 + 0.02z$$

Type II error:

$p = 0.65$

$$p^* = 0.65 - z\sqrt{\frac{(0.65)(0.35)}{900}} \approx 0.65 - 0.02z$$

$0.6 + 0.02z = 0.65 - 0.02z$

$0.05 = 0.04z$

$z = 1.25$

$p^* = 0.6 + 0.02(1.25) \approx 0.63$

D.R. Take a sample N = 900. If $\overline{P} \geq 0.63$ then reject H_o; otherwise reject H_a.

For z = 1.25, from the normal distribution table $\alpha = \beta = 0.5 - 0.3944 \approx 0.11$.

4.

➤a.

We us the estimate $N = \dfrac{z^2 p(1 - p)}{e*^2}$.

For a 95% confidence we have z = 1.96, e = 0. 01 and p = 0.5.

$$N = (1.96)^2 \frac{(0.5)(1 - 0.5)}{(0.01)^2} = 9{,}604.$$

➤b.

Since we are concerned about an excess of defective ball bearings,

$H_o : p = 0.04$

H_a: p > 0.04

➤c.
D.R. Take a sample of size 9,604. If $\bar{P} \geq p^*$ *then reject* H_o *and accept* H_a*; otherwise reject* H_a.

➤d.
Type I error:

$$p^* = 0.04 + z\sqrt{\frac{0.04(0.96)}{9604}} \approx 0.04 + z0.002$$

From the normal distribution table, for the area 0.5 - 0.05 = 0.45, z = 1.64.

p* = 0.04 + (1.64)(0.002) = 0.043.

The decision rule will be:

D.R. Take a sample of size 9,604. If $\bar{P} \geq 0.043$ *then reject* H_o *and accept* H_a*; otherwise reject* H_a.

➤e.
Type II error:

p = 0.06

$$\sigma_{\bar{P}} = \sqrt{\frac{(0.06)(0.94)}{9603}} = 0.002$$

$$z = \frac{0.043 - 0.06}{0.002} = -8.5$$

For such a large value (in absolute terms), β = 0.

5.
➤a.
H_o: p = 0.57
H_a: p ≠ 0.57

➤b. Case 1: Type I error.

Step 1: p = 0.57

$$p^* = z\bar{p} = z\sqrt{\frac{p(1-p)}{N}} = z\sqrt{\frac{(0.57)(1-0.57)}{N}} \approx z\frac{0.55}{\sqrt{N}}$$

Step 2: From the standard normal distribution table, we look-up z for 0.5 - 0.05 = .45 and we find z = 1.64.

$$p^* = z\frac{\sigma}{\sqrt{N}} = (1.64)\frac{0.55}{\sqrt{N}} = \frac{0.902}{\sqrt{N}}$$, equation 1

Case 2: Type II error.

p = 0.51

Since we require $P\{\bar{p} \geq 0.57\} = 0.025$,

$$0.57 - p* = 0.51 + z\frac{\sqrt{(0.51)(0.49)}}{\sqrt{N}} \approx 0.51 + z\frac{0.5}{\sqrt{N}}$$

$$p* = 0.06 - z\frac{0.5}{\sqrt{N}}$$

For $0.51 \leq \bar{p} \leq 0.57 - p^*$ we have $P\{0.51 \leq \bar{p} \leq 0.57 - p^*\} = 0.45$, z = 1.64

$$p* = 0.06 - z\frac{0.5}{\sqrt{N}} = p* = 0.06 - \frac{(1.64)0.5}{\sqrt{N}} = 0.06 - \frac{0.82}{\sqrt{N}}$$, equation 2

Since equation 1 and equation 2 are equal $0.06 - \frac{0.82}{\sqrt{N}} = \frac{0.902}{\sqrt{N}}$

$$\sqrt{N} = \frac{1.722}{0.06} = 28.7$$

$N \approx 824$

$$p^* = \frac{0.902}{\sqrt{N}} = \frac{0.902}{28.7} \approx 0.03$$

➤ c.

D.R. For a sample size N = 824, if $0.57 - 0.03 \leq \bar{p} \leq 0.57 + 0.03$ then reject H_a; otherwise reject H_a and accept H_0

➤d.

$$\text{error} = \bar{p} - p = 0.55 - p = z\sigma_{\bar{p}} = z\sqrt{\frac{(0.55)(0.45)}{824}} \approx 0.02$$

Solving for $z \approx 1.15$

Using the normal distribution table, for z = 1.15, we have the area 0.3749. Therefore, the probability that the error will exceep 2% is 1 - 2(0.3749) = 0.2502.

➤e.
We use the confidence interval: $\bar{p} - z\sigma_{\bar{p}} \le p \le \bar{p} + z\sigma_{\bar{p}}$

$\bar{p} = 0.55$

$$\sigma_{\bar{p}} = \sqrt{\frac{(0.55)(0.45)}{824}} \approx 0.02$$

For a confidence interval of 95%, we use Table C to find z for the area 0.95/2 = 0.475. From the table we find z = 1.96.

From the above confidence formula we have $0.55 - 1.96(0.02) \le p \le 0.55 + 1.96(0.02) =$
$0.5108 \le p \le 0.5892$