Statistical Inference Theory
Lesson 35
The Distribution of $\overline{P}$

35.1 - What is the Central Limit Theorem for $\overline{P}$?

35.1 - Problem 1:

➤(a).

We are given that p = = 0.7 and the sample size taken is n = 100.

From the Central Limit Theorem, we have $\sigma_{\overline{P}} = \sqrt{\frac{p(1-p)}{N}} = \sqrt{\frac{(0.70)(0.3)}{100}} \approx 0.046.$

➤(b).

We use the formula $z = \frac{\overline{P} - p}{\sigma_{\overline{P}}}$ to find the area under the normal distribution curve for

$\overline{P} = 0.65,$

p = 0.70,

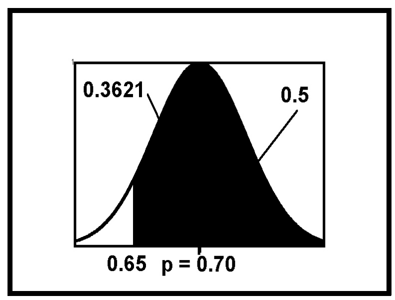
$\sigma_{\overline{P}} = 0.046.$

Therefore, $z = \frac{0.65 - 0.70}{0.046} \approx -1.09.$

From the Normal Distribution tables,

$P\{\overline{P} > 0.65\} = P\{Z > -1.09\} = 0.5 + 0.3621 = 0.8621$

b



➤(c).

From (b). we have $P\{\overline{P} > 0.65\} = 0.8621$.Therefore, $P\{\overline{P} < 0.65\} = 1 - 0.8621 = 0.1379.$

➤(d).

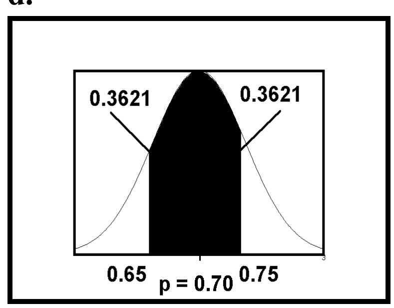
$z = \frac{0.65 - 0.70}{0.046} \approx -1.09.$

From (a). we have, $P\{0.65 \le \overline{P} \le 0.70\} = 0.3621$

Since $P\{0.70 \le \overline{P} \le 0.75\} = P\{0.65 \le \overline{P} \le 0.70\} =$ 0.3621.

Therefore, $P\{0.65 \le \overline{P} \le 0.75\} =$

d.



$P\{0.65 \le \overline{P} \le 0.70\} + P\{0.70 \le \overline{P} \le 0.75\} = 0.3621 + 0.3621 = 0.7242$

35.1 - Problem 2:

➤(a).

We are given that p = 0.5 and the sample size taken is n = 400.

From the Central Limit Theorem, we have $\sigma_{\overline{P}} = \sqrt{\frac{p(1-p)}{N}} = \sqrt{\frac{(0.50)(0.5)}{400}} = 0.025.$

We use the formula $z = \frac{\overline{P} - p}{\sigma_{\overline{P}}}$ to find the area under the normal distribution curve for

$\overline{P} = 0.45$,
p = 0.50,
$\sigma_{\overline{P}} = 0.025$.

Therefore, $z = \frac{0.45 - 0.50}{0.025} \approx -2.$

From the Normal Distribution tables, $P\{\overline{P} \le 0.45\} = P\{Z \le -2\} = 0.5 - 0.4772 = 0.0228$

➤(b).

We can state the decision rule as: A random survey of 400 is taken of its subscribers. If at most c* of its subscribers are men, the company will increase its efforts to have more men subscribe.

$P\{\overline{P} \le c*\} = 0.01$

p = 0.5

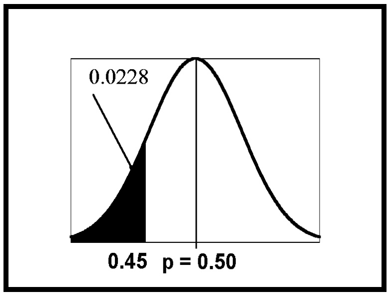
$c^* = p - z\sigma_{\overline{P}} = 0.5 - z0.025$

Form the Normal distribution table, for area 0.5 - 0.01 = 0.49, z = -2.33. Therefore,
c* = 0.5 - 2.33(0.025) = 0.44

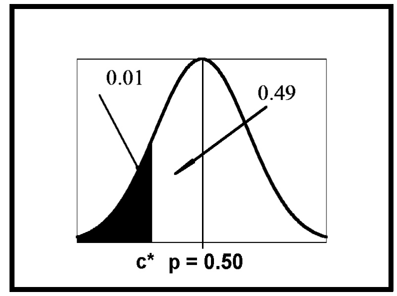
The decision rule can be stated as:

A random survey of 400 is taken of its subscribers. If at most 44% of its subscribers are men, the company will increase its efforts to have more men subscribe.

a.



b.



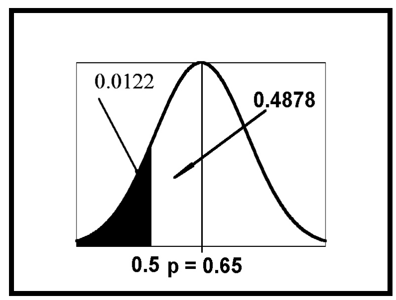
35. - Solving Binomial Problems Using $\overline{P}$

Problem 2.1:

➤(a).

To use the distribution of $\overline{P}$, we need to convert the problem into proportions:

a.



Step 1: p = 0.65.

Step 2:"less than 25 owners" changes to the event $\{\overline{P} < 0.50\}$, where 25/50 = 0.50.

Step 3: $\sigma_{\overline{p}} = \sqrt{\dfrac{p(1-p)}{n}} = \sqrt{\dfrac{0.65(0.35)}{50}} \approx 0.067$

Step 4: $z = \dfrac{0.50 - 0.65}{0.067} = -2.24$

From the table $P\{\overline{P} < 0.50\} = P\{z \le 2.24\}$ = 0.5 - 0.4878 = 0.0122

➤ (b).

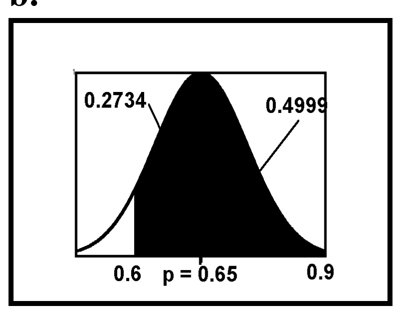
The event "between 30 and 45 " converts to $\{0.6 \le \overline{P} \le 0.9\}$ where 30/50 = 0.6 and 45/50 = 0.9.

Step 2: $z = \dfrac{0.6 - 0.65}{0.067} \approx -0.75$

Step 3: $z = \dfrac{0.9 - 0.65}{0.067} \approx 3.73$

Step 4: From the table,

b.



$P\{0.6 \le \overline{P} \le 0.9\} = P\{-0.75 \le \overline{P} \le 3.73\} = 0.2734 + 0.4999 = 0.7733$

Supplementary Problems

1.

➤a.

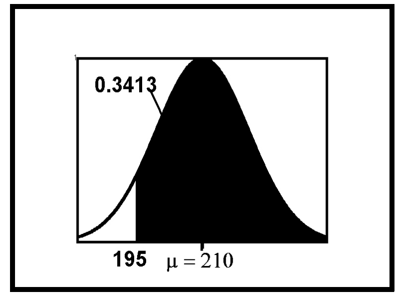
$p = q = 0.5$

$N = 100$

$n = 36$

$$\sigma_{\overline{P}} = \frac{\sqrt{pq}\sqrt{(N - n)}}{\sqrt{n(N - 1)}} = \frac{\sqrt{0.5(0.5)}\sqrt{100 - 36}}{\sqrt{36(100 - 1)}} \approx 0.067$$

navy officers



➤b.

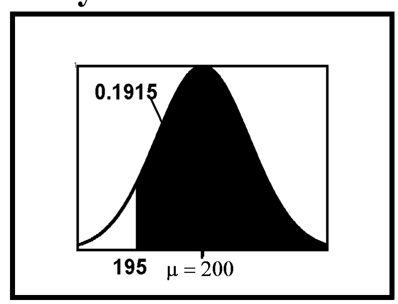
$p = 0.5$

We need to find $P\{\overline{P} \ge 0.60\}$.

$$z = \frac{0.6 - 0.5}{0.067} \approx 1.49$$

$P\{\overline{P} \ge 0.60\} = P\{Z \ge 1.49\} = 5. - 0.4319 = 0.0681$

army officers



2.

➤a.

Step 1: We find the proportion of army officers that weigh more than 195 pounds

$$z = \frac{195 - 200}{10} \approx -0.5$$

$P\{X \ge 195\} = P\{Z \ge -0.5\} = 0.5 + 0.1915 = 0.6915$

$p \approx 0.69$, the proportion of army officers that weigh more than 195 pounds.

We find the proportion of navy officers that weigh more than 195 pounds.

$$z = \frac{195 - 210}{15} \approx -1$$

$P\{X \ge 195\} = P\{Z \ge -1\} = 0.5 + 0.3413 = 0.8413$

$p \approx 0.84$, the proportion of navy officers that weigh more than 195 pounds.

Step 2: For the army officers we need to find $P\{\overline{P} \geq 0.75\}$,

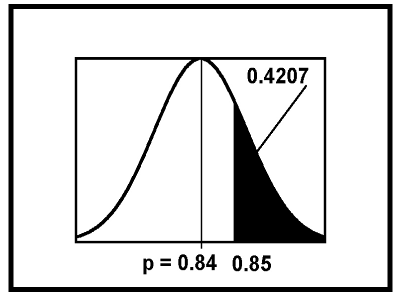
where p = 0.69 and N = 36.

$$\sigma_{\overline{P}} = \sqrt{\frac{0.69(0.31)}{36}} \approx 0.08$$

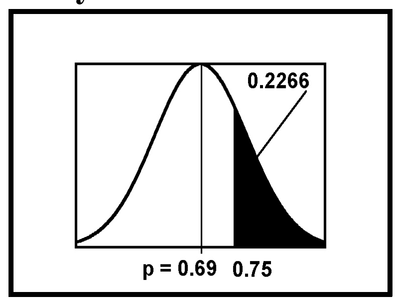
$$z = \frac{0.75 - 0.69}{0.08} = 0.75$$

$$P\{\overline{P} \geq 0.75\} = P\{Z \geq 0.75\} = 0.5 - 0.2734 = 0.2266$$

navy officers



army officers



Step 3: For the navy officers we need to find $P\{\overline{P} \geq 0.85\}$,
where p = 0.84 and N = 49.

$$\sigma_{\overline{P}} = \sqrt{\frac{0.84(0.16)}{49}} \approx 0.05$$

$$z = \frac{0.85 - 0.84}{0.05} = 0.2$$

$$P\{\overline{P} \geq 0.85\} = P\{Z \geq 0.2\} = 0.5 - 0.0793 = 0.4207$$

Step 4: The probability that at least 75% of the army officers weigh more than 195 pounds and at least 85% of the naval officers weigh more than 195 pounds is (0.2266)(0.4207) ≈ 0.10.

➤b.
Here we add the two probabilities: 0.2266 + 0.4207 = 0.6473

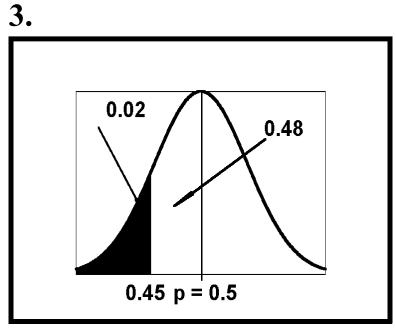
3.
p = 0.50

$$\sigma_{\overline{p}} = \sqrt{\frac{(0.5)(0.5)}{N}} = \frac{0.5}{\sqrt{N}}$$

$$p* = p + z\sigma_{\overline{p}}$$

$$0.45 = 0.5 + z\frac{0.5}{\sqrt{N}}$$

3.



For the area 0.48, from the normal distribution table, z = -2.05

Therefore, $0.45 = 0.5 + z\frac{0.5}{N} = 0.5 - \frac{(2.05)(0.5)}{\sqrt{N}}$

Solving the equation for N we have $\sqrt{N} = 20.5$.

$N \approx 421$

4..

$p = \overline{P} - z\sigma_{\overline{P}} = \overline{P} - z\sqrt{\frac{p(1-p)}{N}}$

From the normal distribution table: $p = 0.35 + 2.06\sqrt{\frac{p(1-p)}{N}}$,

$p = 0.45 - 2.33\sqrt{\frac{p(1-p)}{N}}$

Solving for p:

$$\sqrt{\frac{p(1-p)}{N}} = \frac{p - 0.35}{2.06}$$

$$\sqrt{\frac{p(1-p)}{N}} = \frac{-p + 0.45}{2.33}$$

$$\frac{p - 0.35}{2.06} = \frac{-p + 0.45}{2.33}$$

$p \approx 0.40$

$$\sqrt{\frac{(0.4)(0.6)}{N}} = \frac{0.4 - 0.35}{2.06}$$

$N \approx 420$

5.

➤a.

Since her decision is a result of tossing a fair coin, her chance of winning a game is p = 0.5.

E(X) = 0.5(\$100) - 0.5(\$110) = -\$5.00 per game.

Since is will play 90 games per month, E(90X) = 90(-\$5.00) = -\$450 per month.

Since is will play 9(90) = 270 games for the entire season, E(270X) = 270($5.00) = -$1,350.

➤b.
The following partial table is the amount that can be lost at the end of each month:

THE NUMBER OF LOST GAMES IN A MONTH	LOSSES
43	-$30
44	-$240
...	...

Therefore, for any given month, to lose is $P\{X \geq 48\} = P\{\overline{P} \geq \frac{43}{90}\} \approx P\{\overline{P} \geq 0.48\}$.

$$\sigma_{\overline{P}} = \sqrt{\frac{(0.5)(0.5)}{90}} \approx 0.05$$

$$z = \frac{0.48 - 0.50}{0.05} = -0.4.$$

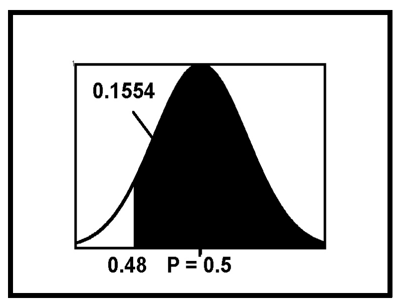
From the normal distribution table, for z = -0.4

$P\{\overline{P} \geq 0.48\} = P\{Z \geq -0.4\} = 0.1554 + 0.5 = 0.6554$, the probability that she will lose money for any given month.

➤c.
This is a binomial problem. Here, $p \approx 0.66$, N = 9 and k = 6.

$$\binom{9}{6}(0.66)^6(0.34)^3 \approx 0.27$$

b.



6.
➤(a).
N = 200
p = 0.55

$$\sigma_{\overline{P}} = \sqrt{\frac{(0.55)(0.45)}{200}} = 0.035$$

We need to find $P\{\overline{P} \leq 0.53\}$.

$$z = \frac{0.53 - 0.55}{0.035} = -0.57$$

$P\{\overline{P} \leq 0.53\} = P\{Z \leq -0.57\} = 0.5 - 0.2157 \approx 0.28$

➤(b).
N = 200

p = 0.5

$$\sigma_{\overline{P}} = \sqrt{\frac{(0.5)(0.5)}{200}} = 0.035$$

$P\{\overline{P} \geq 0.53\}$

$$z = \frac{0.53 - 0.5}{0.035} = 0.86$$

$P\{\overline{P} \geq 0.53\} = P\{Z \geq 0.86\} = 0.5 - 0.3051 \approx 0.19$

➤(c).
We restate the decision rule:

If at least p% of the voters say they are in favor of free trade, she will state in her election advertisements that she supports free trade. However, if less than p% say they are in favor of free trade, she will state in her election advertisements that she does not support free trade.

N = 200
p = 0.5

$$\sigma_{\overline{P}} = \sqrt{\frac{(0.5)(0.5)}{200}} = 0.035$$

$P\{\overline{P} \geq p\%\} = 0.01$

$p\% = p + z\sigma_{\overline{P}} = 0.5 + z(0.035)$

For the area 0.5 - 0.01, z = 2.33.

$p\% = p + z\sigma_{\overline{P}} = 0.5 + 2.33(0.035) = 0.58$.

We state the decision rule:

If at least 58% of the voters say they are in favor of free trade, she will state in her election advertisements that she supports free trade. However, if less than 58% say they are in favor of free trade, she will state in her election advertisements that she does not support free trade.

➤d.

p = 0.55

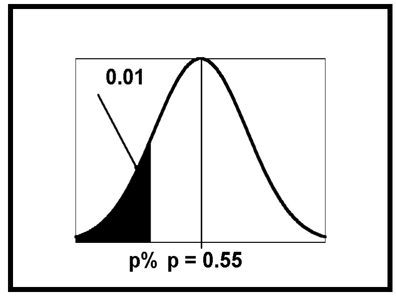
$$\sigma_{\overline{P}} = \sqrt{\frac{(0.55)(0.45)}{N}} \approx \frac{0.5}{\sqrt{N}}$$

$P\{\overline{P} \le p\%\} = 0.01$

For the area 0.5 - 0.01 = 0.49, z = -2.33

$p\% = p + z\sigma_{\overline{P}} = 0.55 + -2.33\sqrt{\frac{0.25}{N}}$, equation A.

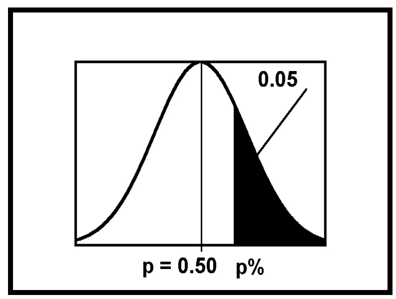
d



p = 0.5

$$\sigma_{\overline{P}} = \sqrt{\frac{(0.5)(0.5)}{N}} \approx \frac{0.5}{\sqrt{N}}$$

d



$P\{\overline{P} \ge p\%\} = 0.05$

For the area 0.5 - 0.0 = 0.45, z = 1.64

$p\% = p + z\sigma_{\overline{P}} = 0.5 + 1.64\frac{0.5}{\sqrt{N}}$, equation B.

Setting equations A and B equal, we have

$$0.55 + -2.33\sqrt{\frac{0.25}{N}} = 0.5 + 1.64\frac{0.5}{\sqrt{N}}$$

$$\frac{0.05}{2.33} = \frac{0.5}{\sqrt{N}} + \frac{0.5}{\sqrt{N}} = \frac{1}{\sqrt{N}}$$

$N \approx 2{,}172$

$$p\% = 0.5 + 1.64\frac{0.5}{\sqrt{N}} = 0.5 + 1.64\frac{0.5}{46.6} \approx 0.52.$$

Take a sample of size 2,172. If at least 52% of the voters say they are in favor of free trade, she will *state in her election advertisements that she supports free trade. However, if less than 52% say they are in favor of free trade, she will state in her election advertisements that she does not support free trade.*

7.

$P\{X_k = 1\} = p$ and $P\{X_k = 0\} = 1 - p = q$ on the kth trial ($k = 1, 2, 3,..., n$). Show

➤a.
$P\{X_k = 1\} = p$ and $P\{X_k = 0\} = 1 - p = q$ on the kth trial ($k = 1, 2, 3,..., n$).

$E(X_k) = 1P\{X_k = 1\} + 0P\{X_k = 0\} = 1p = p$

➤ b.
$E(X^2_k) = 1^2P\{X_k^2 = 1^2\} + 0^2P\{X_k^2 = 0^2\} = 1P\{X_k^2 = 1\} + 0P\{X_k = 0\} = 1p = p$

$\sigma^2_{Xk} = E(X_k^2) - [E(X_k)]^2 = p - p^2 = p(1 - p) = pq$

➤ c.
$E(X_k/N) = p/N$

$E[(X_k/N)^2] = \sigma^2_{X_1 + X_2 + ... + X_n} = \sigma^2_{X_2} + \sigma^2_{X_2} + ... + \sigma^2_{X_n}$

$\sigma^2_{Xk/N} = p(1 - p)/N^2$

$$\sigma^2_{\overline{P}} = \sigma^2_{\frac{X_1 + X_2 + ... + X_N}{N}} = \sigma^2_{\frac{X_1}{N}} + \sigma^2_{\frac{X_2}{N}} + ... + \sigma^2_{\frac{X_N}{N}} = \frac{p(1-p)}{N^2} + ... + \frac{p(1-p)}{N^2} = p(1-p)/N$$

$$\sigma_{\overline{P}} = \sqrt{\frac{p(1-p)}{N}}$$

8.
From Lesson 16, problem 13, we have $E(S^2) = \frac{n-1}{n}\sigma^2$.

For the Bernoulli experiment with n independent trials, we have $\sigma^2 = pq$.

Therefore, $E(S^2) = \frac{n-1}{n}pq$

9.

➤ a.

See lesson 27, 27.1-Example 1, (e). $P\{45 \le X \le 55\} \approx 0.7286$

➤ b.

$$\sigma_{\overline{P}} = \sqrt{\frac{0.5(1 - 0.5)}{100}} = 0.05$$

$$z = \frac{0.55 - 0.5}{0.05} = 1$$

$$z = \frac{0.45 - 0.5}{0.05} = -1$$

$P\{45 \le X \le 55\} = P\{-1 \le Z \le 1\} \approx 0.3413 + 0.3413 = 0.6826$