Statistical Inference Theory
Lesson 34
Combining Type I and Type II Errors

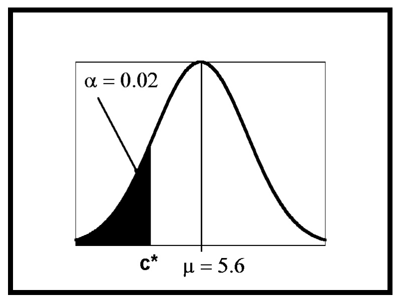
34.1 - Creating Decision Rules for a one-sided test from α and β.

34.1 - Problem 1:

➤(a).

Since we wish to test the claim that the new laser machine will decrease the average production time we make H_0 claim that there is no change and H_a the counter-claim that the laser machine does reduce production time.

Case 1



H_o: $\mu = 5.6$

H_a: $\mu < 5.6$

➤(b).

Case 1: Type I error.

Step 1: $\mu = 5.6$

$$c^* = \mu + z\sigma_{\bar{x}} = 5.6 + z\frac{\sigma}{\sqrt{N}} = 5.6 + z\frac{0.45}{\sqrt{N}}$$

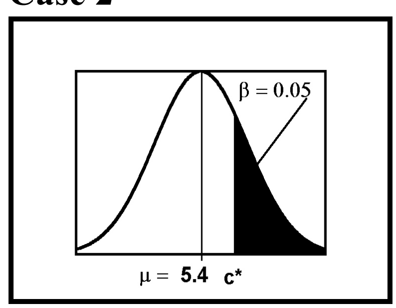
Step 2: $\alpha = P\{\bar{X} < c^* < 5.6\} = 0.02$

From the standard normal distribution table, we look-up z for 0.5 - 0.02 = 0.48: z = - 2.05.

$$c^* = 5.6 + z\frac{0.45}{\sqrt{N}} = 5.6 - (2.05)\frac{0.45}{\sqrt{N}} = 5.6 - \frac{0.9225}{\sqrt{N}}$$

$$c^* = 5.6 - \frac{0.9225}{\sqrt{N}}$$,equation 1.

Case 2



Case 2: Type II error.

Step 1: $\mu = 5.4$

$c^* = \mu + z\sigma_{\bar{x}} = 5.4 + z\frac{\sigma}{\sqrt{N}} = 5.4 + z\frac{0.45}{\sqrt{N}}$

Step 2: $\beta = P\{5.4 < c^* < \overline{X}\} = 0.05$

From the standard normal distribution table, we look-up z for 0.5 - 0.05 = .45: z = 1.64.

$$c^* = 5.4 + z\frac{0.45}{\sqrt{N}} = 5.4 + (1.64)\frac{0.45}{\sqrt{N}} = 5.4 + \frac{0.738}{\sqrt{N}}$$

$$c^* = 5.4 + \frac{0.738}{\sqrt{N}}$$, equation 2.

Since the c* and N are the same for equation 1 and equation 2, we set the two equations equal and solve first for N:

$$5.6 - \frac{0.9225}{\sqrt{N}} = 5.4 + \frac{0.738}{\sqrt{N}}$$

$$5.6\sqrt{N} - 0.9225 = 5.4\sqrt{N} + 0.738$$

$$0.2\sqrt{N} = 1.6605$$

$$\sqrt{N} = 8.3025$$

$$N \approx 69$$

From equation 2, $c^* = 5.4 + \frac{0.738}{\sqrt{N}} = 5.4 + \frac{0.738}{8.3025} \approx 5.49$

➤(c).
In the above decision rule, replace c* with 5.49 and N with 69:

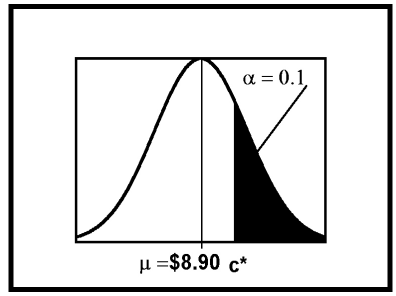
*Decision Rule: Take a sample of 69 transmissions. If $\overline{X} < 5.49$, then **reject H_o and accept H_a**; otherwise, **reject H_a**.*

34.1- Problem 2:
➤(a).
We make the union claim H_0: μ = \$8.90 and the counter-claim as H_a: μ > \$8.90.

➤(b).

Case 1: Type I error.

Case 1



Step 1: $\mu = \$8.90$

$$c^* = \mu + z\sigma_{\bar{x}} = 8.9 + z\frac{\sigma}{\sqrt{N}} = 8.9 + z\frac{1}{\sqrt{N}}$$

Step 2: $\alpha = P\{8.9 < c^* < \bar{X}\} = 0.1$

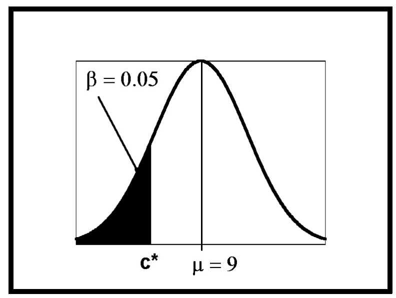
From the standard normal distribution table, we look-up z for 0.5 - 0.1 = 0.4: z = 1.28

$$c^* = 8.9 + z\frac{1}{\sqrt{N}} = 8.9 + \frac{1.28}{\sqrt{N}}$$

$$c^* = 8.9 + \frac{1.28}{\sqrt{N}}$$,equation 1.

Case 2: Type II error.

Case 2



Step 1: $\mu = \$9$

$$c^* = \mu + z\sigma_{\bar{x}} = 9 + z\frac{\sigma}{\sqrt{N}} = 9 + \frac{z}{\sqrt{N}}$$

Step 2: $\beta = P\{\bar{X} < c^* < 9\} = 0.05$

From the standard normal distribution table, we look-up z for 0.5 - 0.05 = .45: z = -1.64.

$$c^* = 9 + \frac{z}{\sqrt{N}} = 9 - \frac{1.64}{\sqrt{N}}$$

$$c^* = 9 - \frac{1.64}{\sqrt{N}}$$, equation 2.

Since the c* and N are the same for equation 1 and equation 2, we set the two equations equal and solve first for N:

$$9 - \frac{1.64}{\sqrt{N}} = 8.9 + \frac{1.28}{\sqrt{N}}$$

$$9\sqrt{N} - 1.64 = 8.9\sqrt{N} + 1.28$$

$$(0.1)\sqrt{N} = 2.92$$

$\sqrt{N} = 29.2$

$N = 853$

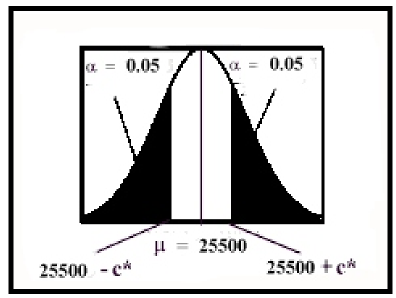
Using equation 2, $c^* = 9 - \frac{1.64}{\sqrt{N}} = 9 - \frac{1.64}{29.2}$ =\$ 8.94

➤(c).
Decision Rule: *For N = 853, If* $\bar{x} > \$8.94$*, then reject* H_o *and accept* H_a*; otherwise, reject* H_a*.*

34.2 - Creating Decision Rules for a two-sided test from α and β.

34.2 - Problem 1:
➤(a).
H_o: $\mu = 25{,}500$
H_a: $\mu \neq 25{,}500$

Case 1



➤(b.)
Case 1: Type I error.

Step 1: $\mu = 25{,}500$

$$c^* = z\sigma_{\bar{x}} = z\frac{\sigma}{\sqrt{N}} = z\frac{1000}{\sqrt{N}}$$

Step 2: From the standard normal distribution table, we look-up z for 0.5 - 0.05 = .45 and we find z = 1.64.

$$c^* = z\frac{\sigma}{\sqrt{N}} = (1.64)\frac{1000}{\sqrt{N}} = \frac{1640}{\sqrt{N}}$$, equation 1

Case 2: Type II error.

$\mu = 27{,}700$

Since we require $P\{X < 25{,}500\} = \frac{\beta}{2} = 0.005$,

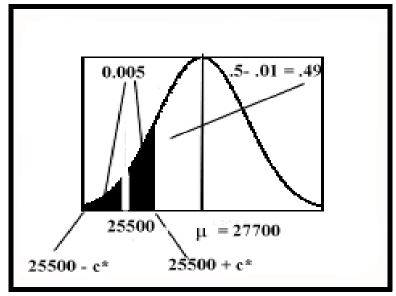
$$25500 + c* = 27700 + z\frac{\sigma}{\sqrt{N}} = 27700 + z\frac{1000}{\sqrt{N}}$$

$$c* = 2200 + z\frac{2000}{\sqrt{N}}$$

For $25500 + c^* < \overline{X} < 27700$, we have $P\{25500 + c^* < \overline{X} < 27700\} = 0.49$, $z = -2.33$

$$c^* = 200 + (-2.33)\frac{1000}{\sqrt{N}} = 200 - \frac{2330}{\sqrt{N}}, \text{ equation 2}$$

Case 2



Since equation 1 and equation 2 are equal

$$\frac{1640}{\sqrt{N}} = 200 - \frac{2330}{\sqrt{N}}$$

$$\frac{3970}{\sqrt{N}} = 200$$

$$\sqrt{N} = \frac{3970}{200} = 19.50$$

$N \approx 380$

$$c^* = z\frac{\sigma}{\sqrt{N}} = (1.64)\frac{1000}{\sqrt{N}} = \frac{1640}{\left(\frac{3970}{200}\right)} \approx 82.62$$

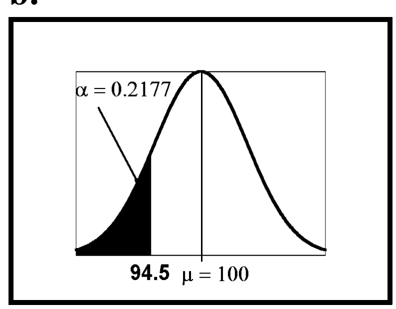
➤(c).
D.R.: Take a sample of size N = 380. If 25,500 -82.62 ≤ $\overline{X}$ *≤ 25,500 + 82.62, then* ***reject*** H_a***;*** *otherwise,* ***reject*** H_0 ***and accept*** H_a***.***

Supplementary Problems

1.
➤a.
We start with the claim that at least 50% of the voters support free trade: $p \geq 0.5$. The counter-claim is $p < 0.5$. Therefore,

H_o: $\mu = 200(0.5) = 100$
H_a: $\mu < 100$

b.



➤b.
$\mu = 100$

$$\sigma = \sqrt{200(0.5)(0.5)} \approx 7.07$$

$\alpha = P\{X \leq 94.5\}$

$$z = \frac{94.5 - 100}{7.07} \approx -0.78$$

From the normal distribution table, $\alpha = P\{X \le 94.5\}$ $\alpha = P\{Z \le -0.78\} = 0.5 - 0.2823 \approx 0.2177$

➤c.

p = 0.4

$\mu = 0.4(200) = 80$

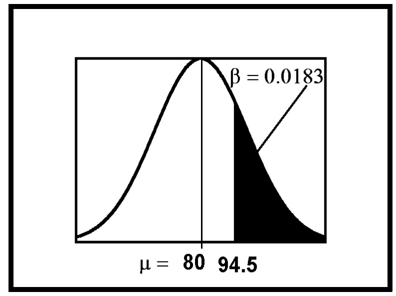
$$\sigma = \sqrt{200(0.4)(0.6)} \approx 6.93$$

$$z = \frac{94.5 - 80}{6.93} \approx 2.09$$

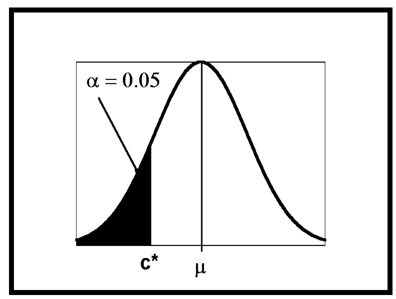
From the normal distribution table,

$\beta = P\{X \ge 94.5\} = P\{Z \ge 2.09\} = 0.5 - 0.4817 = 0.0183$

c.



Case 1



➤d.

Case 1: Type I error, $\alpha = 0.05$

Step 1: $\mu = 0.5N$

$$\sigma = \sqrt{N(0.5)(0.5)} \approx 0.5\sqrt{N}$$

Step 2: $\alpha = P\{\overline{X} < c^* < \mu\} = 0.05$

From the standard normal distribution table, we look-up z for 0.5 - 0.05 = 0.45: z = - 2.05.

$$c^* = 0.5N + z\sigma = 0.5N - (1.64)(0.5)\sqrt{N} = 0.5N - 0.82\sqrt{N}$$, equation 1.

Case 2: Type II error, $\beta = 0.01$

Step 1: From the standard normal distribution table, we look-up z for 0.5 - 0.01 = 0.49: z = 2.33

$\beta = P\{\mu < c^* < \overline{X}\} = 0.01$

$\mu = 0.4N$

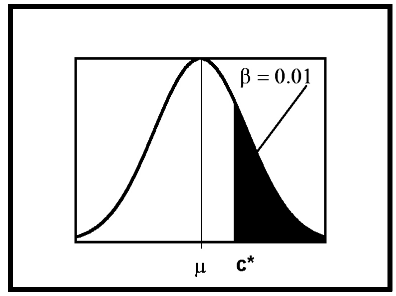
$$\sigma = \sqrt{N(0.4)(0.6)} \approx 0.49\sqrt{N}$$

$$c^* = 0.4N + z\sigma = 0.4N + (2.33)(0.49)\sqrt{N} \approx 0.4N + 1.14\sqrt{N}$$, equation 2.

Step 2: Since the c* and N are the same for equation 1 and equation 2, we set the two equations equal and solve first for N:

$$0.5N - 0.82\sqrt{N} = 0.4N + 1.14\sqrt{N}$$

Case 2



$$1.96\sqrt{N} = (0.1)N$$

$$\sqrt{N} \approx 19.6$$

$$N \approx 400$$

From equation 2,

$$c^* = 0.4N + 1.14\sqrt{N} \approx (0.4)400 + 1.14(19.6) \approx 182.34$$

Therefore, the decision rule would be:

D.R. A sample of size 400 is taken. If at least 183 of the voters say they are in favor of free trade, she will state in her election advertisements that she supports free trade. However, if less than 183 say they are in favor of free trade, she will state in her election advertisements that she does not support free trade.

2.

➤a.

Since we wish to test the belief that a national bank discriminates, we make the counter-claim $p < 0.17$, below the proportion that this particular group represents in the population. We make the claim that there is no discrimination: $p = 0.17$. Therefore,

$H_o : \mu \geq 17$
$H_a : \mu < 17$

➤b.

$p = 0.17$
$\mu = 17$

$$z = \sqrt{100(0.17)(0.83)} \approx 3.76$$

From the decision rule, $\alpha = P\{X \leq 16.5\}$.

$$z \approx \frac{16.5 - 17}{3.76} = -0.13$$

From the normal distribution table, $\alpha = P\{X \leq 16.5\} = P\{Z \leq -0.13\} = 0.5 - 0.0517 = 0.4483$.

➤c.
p = 0.12

$\mu = 100(0.12) = 12$

$z = \sqrt{100(0.12)(0.88)} \approx 3.25$

From the decision rule, $\beta = P\{X \geq 16.5\}$.

$$z \approx \frac{16.5 - 12}{3.25} = 1.38$$

From the normal distribution table, $\beta = P\{X \geq 16.5\} = P\{Z \geq 1.38\} = 0.5 - 0.4162 = 0.0838$.

➤d.
Case 1: Type I error, $\alpha = 0.02$

Step 1: p = 0.17

$\mu = 0.17N$

$\sigma = \sqrt{N(0.17)(0.83)} \approx 0.38\sqrt{N}$

Step 2: $\alpha = P\{\overline{X} < c^* < \mu\} = 0.02$

From the standard normal distribution table, we look-up z for 0.5 - 0.02 = 0.48: z = - 2.05.

$c^* = 0.17N + z\sigma = 0.17N - (2.05)(0.38)\sqrt{N} = 0.17N - 0.78\sqrt{N}$, equation 1.

Case 2: Type II error, $\beta = 0.01$

Step 1: p = 0.12

From the standard normal distribution table, we look-up z for 0.5 - 0.05 = 0.45: z = 1.64

$\beta = P\{\mu < c^* < \overline{X}\} = 0.05$

$\mu = 0.12N$

$\sigma = \sqrt{N(0.12)(0.88)} \approx 0.32\sqrt{N}$

$c^* = 0.12N + z\sigma = 0.12N + (1.64)(0.32)\sqrt{N} \approx 0.12N + 0.52\sqrt{N}$, equation 2

Step 2: Since the c* and N are the same for equation 1 and equation 2, we set the two equations equal and solve first for N:

$0.12N + 0.52\sqrt{N} = 0.17N - 0.78\sqrt{N}$

$1.3\sqrt{N} = (0.05)N$

$1.3/0.05 = \sqrt{N} = 26$

$N = 676$

From equation 2, $c^* = 0.12N + 0.52\sqrt{N} = 0.12(676) + (0.52)26 \approx 95$

Therefore, the decision rule would be:

D.R. A sample of size 676 loans is taken. If at least 95 of the loans have been given to the minority group then the Federal agency will reserve judgement. However, if less than 95 of the loans have been given to the group than the agency will conclude that the bank is discriminating against this minority group.