Statistical Inference Theory
Lesson 33
Type II Errors

33.1- Two-sided test

33.1 - Problem 1:

➤(a).

Since the sample average $\bar{x} = 117.25$ is inside the interval $115 \le \bar{x} \le 135$,

the decision rule requires that you reject H_a.

➤(b).

Step 1: Since the population average $\mu = 131$, the claim H_o: $\mu = 125$ is false and the counter-claim H_a: $\mu \ne 125$ is true.

Step 2: Since the sample average $\bar{x} = 117.25$ is inside the interval $115 \le \bar{x} \le 135$, the decision rule requires that you reject H_a.

Step 3: Since you are rejecting H_a, which is true, a Type II error occurred.

➤(c).

Step 1: Since the population average $\mu = 125$, the claim H_o: $\mu = 125$ is true and the counter-claim: H_a: $\mu \ne 125$ is false.

Step 2: Since the sample average $\bar{x} = 117.25$ is inside the interval $115 \le \bar{x} \le 135$, the decision rule requires that you reject H_a .

Step 3: Since you are rejecting H_a, which is false, no error has occurred.

33.1 - Problem 2:

➤(a).

To test any significant changes, we set $\mu = 25500$ and to test against this claim $\mu \ne 25500$. Therefore,

H_o: $\mu = 25{,}500$

H_a. $\mu \ne 25{,}500$

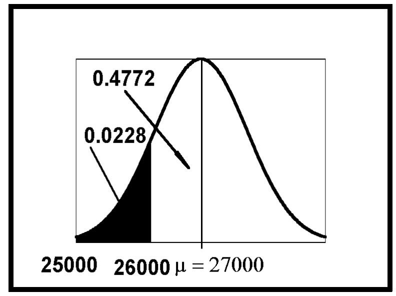
➤(b).
Since $\mu = 27{,}000$, we need to find $\beta = P\{25{,}000 \le \overline{X} \le 26{,}000\}$.

$$\sigma_{\overline{X}} = \frac{5000}{\sqrt{100}} = 500.$$

$$z = \frac{25000 - 27000}{500} = -4$$

$$z = \frac{26000 - 27000}{500} = -2$$

b.



From the normal distribution table, for z = -4 and -2, we have $\beta = P\{25{,}000 \le \overline{X} \le 26{,}000\} = P\{-4 \le Z \le -=2\} \approx 0.5 - 0.4772 = 0.0228$.

➤(c).
Step 1: Since $\mu = 26{,}125$,

H_o is false ($\mu = 25{,}500$) and H_a is true.

Step 2: Since $\overline{x} = 25{,}200$ lies in the interval

$25{,}000 \le \overline{X} \le 26{,}000$,

the decision rule requires us to reject H_a.

Step 3: Since we are rejecting H_a which is true, a Type II error occurs.

➤(d).
Step 1: ***D.R.*** *If 25,500 - c* ≤* $\overline{X}$ *≤ 25,500 + c* then* ***reject*** $\boldsymbol{H_a}$*; otherwise* ***reject*** $\boldsymbol{H_0}$ ***and accept*** $\boldsymbol{H_a}$***.***

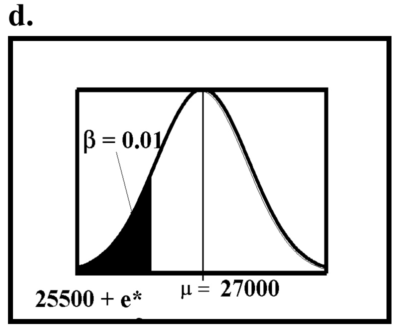
Step 2: Since we assume $\mu = 27{,}000$ and $\beta = 0.01$, set

$P\{25500 + c^* \le \overline{X} \le 27{,}000\} = 0.49$

Step 3: $25500 + c^* = 27000 + Z\sigma_{\overline{X}} = 27000 - 2.33(500)$

$c^* = 335$

d.



Note: $P\{25500 - 335 \le \overline{X} \le 27000\} =$

$P\{25165 \le \overline{X} \le 27000\} = P\{-3.67 \le Z \le 0\} \approx 0.5$

Therefore, the decision rule is: *If 25,165 ≤ $\overline{X}$ ≤ 25,835 then reject H_a; otherwise reject H_0 and accept H_a.*

➤(e).
Step 1: For computing a Type I error, we have μ = 25,500 and the new decision rule:

If 25,165 ≤ $\overline{X}$ ≤ 25,835 then ***reject H_a****; otherwise* ***reject H_0 and accept H_a.***

Step 2:

$$Z = \frac{25165 - 25500}{500} \approx -0.67$$

$$Z = \frac{25835 - 25500}{500} \approx 0.67$$

P{25165 ≤ $\overline{X}$ ≤ 25835} =

P{0 ≤ Z ≤ 0.67} + P{-0.67 ≤ Z≤ 0} = 2P{0 ≤ Z ≤ 0.67 } = 2(0.2486) = 0.4972.

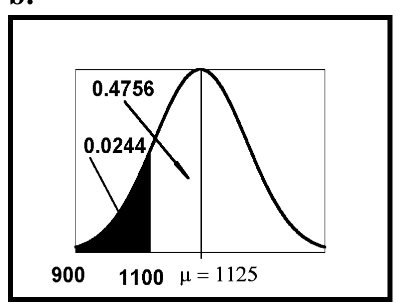
Step 3: α = 1 - 0.4972 ≈ 0.5

33.1 - Problem 3:
➤(a).
Since the claim is 1,000 electric sockets a minute, we have μ = 1,000 and therefore,

H_o: μ = 1,000
H_a.: μ ≠ 1,000

➤(b.)
Since μ = 1125, we need to find β = P{900 ≤ $\overline{X}$ ≤ 1100}.

b.



$$\sigma_{\overline{X}} = \frac{76}{\sqrt{36}} \approx 12.67$$

$$z = \frac{900 - 1125}{12.67} = -17.75$$

$$z = \frac{1100 - 1125}{12.67} = -1.97$$

From the normal distribution table, for z = -1.97 and -17.75, we have

$\beta = P\{900 \le \overline{X} \le 1100\} = P\{-17.75 \le Z \le -1.97\} \approx 0.5 - 0.4756 = 0.0244.$

➤(c).
Step 1: Since $\mu = 1{,}021$, H_o is false ($\mu = 1{,}000$) and H_a is true.

Step 2: Since $\overline{x} = 1{,}121$ outside the interval $900 \le \overline{X} \le 1{,}100$, the decision rule requires us to reject H_o and accept H_a.

Step 3: Since we are rejecting H_0 which is false, and accept H_a is true, no error occurs.

33.2 - One-sided test

33.2 - Problem 1:
➤(a).
Step 1: The alternative to H_a is H_o .

Step 2: Since H_a: $\mu > 125$, the alternative to $\mu > 125$ is $\mu \le 125$.

Step 3: Since H_o is the alternative to H_a, the other version for H_o is H_o: $\mu \le 125$.

➤(b).
Since the sample average $\overline{x} = 120.20 < 128$, the decision rule requires that you reject H_a .

➤(c).
Step 1: Since the population average $\mu = 139.71$, the claim H_o: $\mu \le 125$ is false and the counter-claim H_a: $\mu > 125$ is true.

Step 2: Since the sample average, $\overline{x} = 120.20 < 128$, the decision rule requires that you reject H_a.

Step 3: Since you are rejecting H_a, which is true, a Type II error has occurred.

➤(d).
Step 1: Since the population average $\mu = 100$, the claim H_o: $\mu \le 125$ is true and the counter-claim :H_a: $\mu > 125$ is false.

Step 2: Since the sample average $\overline{x} = 120.20 < 128$, the decision rule requires that you reject H_a .

Step 3: Since you are rejecting H_a , which is false, no error has occurred.

33.2 - Problem 2:

➤(a).

Since we are interested in determining if the new last machine will decrease the average production time, we have $\mu < 5.6$. Therefore,

H_o: $\mu = 5.6$

H_a: $\mu < 5.6$

➤(b).

From the decision rule, if $\mu = 5$, $\beta = P\{\overline{X} \geq 5\}$.

$$\sigma_{\overline{X}} = \frac{0.45}{\sqrt{400}} = 0.0225$$

$$z = \frac{5 - 5}{0.0225} = 0$$

Therefore, $\beta = P\{\overline{X} \geq 5\} = 0.5$

➤(c).

Step 1: Since $\mu = 5.5$, H_a is true.

Step 2: *Decision Rule: If* $\overline{X}$ *< 5, then* ***reject*** H_o ***and accept*** H_a*; otherwise,* ***reject*** H_a**.**

Step 3: Since $\overline{x} = 5.1$, the decision rule requires us to reject H_a. Since we are rejecting H_a, which is true, a Type II error occurs.

➤(d).

we restate the decision rule as: If $\overline{X} < c^*$, then **reject** H_o **and accept** H_a; otherwise, **reject** H_a.

We need to find $\beta = P\{\overline{X} \geq c^*\}$.

$c^* = 5 + z\,(0.0225)$

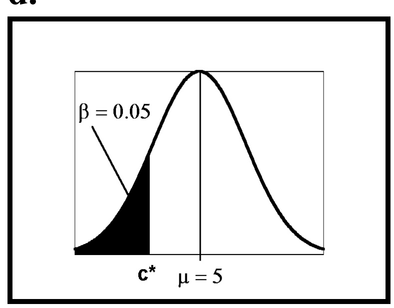
From the normal distribution table, for the area 0.5 - 0.05 = 0.45, z = 1.64.

$c^* = 5 + 1.64\,(0.0225) \approx 5.04$

The decision rule now reads,

Decision Rule: If $\overline{X} < 5.04$*, then reject* H_o *and accept* H_a*; otherwise, reject* H_a*.*

d.



➤(e).
For the type I error, we assume $\mu = 5.6$.

$$z = \frac{5.04 - 5.6}{0.0225} \approx -25$$

Therefore, $\alpha = 0$.

33.2 - Problem 3:
➤(a).
The claim of the union is that H_o: $\mu \leq \$8.90$.

The counter-claim is H_a: $\mu > \$8.90$.

➤(b).
From the decision rule, $\beta = P\{\overline{X} \leq \$9\}$ if $\mu = 9.12$.

$$\sigma_{\overline{X}} = \frac{1}{\sqrt{100}} = 0.1$$

$$z = \frac{9 - 9.12}{0.1} = -1.2$$

From the normal distribution table, for z = -1.2, the area is 0.3849. Therefore,

$\beta = P\{\overline{X} \geq \$9\} = 0.5 - 0.3849 \approx 0.12$.

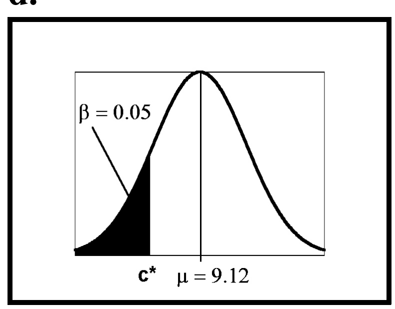
➤(c).
Step 1: Since $\mu = \$9.10$, H_a is true.

Step 2: *Decision Rule: If* $\overline{X} > \$9.00$*, then* ***reject*** H_o *and accept* H_a*; otherwise,* ***reject*** H_a**.**

Since $\overline{x} = \$9.21$, the decision rule requires us to reject H_o and accept H_a.

Since we are accepting H_a , which is true, no error occurs.

d.



➤(d).
we restate the decision rule as:

Decision Rule: If $\overline{X} > c^*$*, then* ***reject*** H_o ***and accept*** H_a***;*** *otherwise****, reject*** H_a**..**

We need to find $\beta = P\{\overline{X} \le c^*\}$.

$c^* = 9.12 - z\,(0.1)$

From the normal distribution table, for the area 0.5 - 0.05 = 0.45, z = 1.64.

$c^* = 9.12 - 1.64\,(0.1) \approx \$8.96$

The decision rule now reads: *If* $\overline{X} > \$8.96$*, then* ***reject*** H_o *and accept* H_a*; otherwise,* ***reject*** H_a.

➤(e).

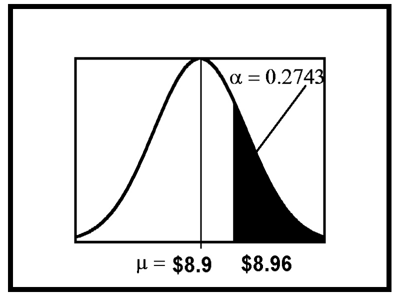
To determine the type I error, we assume $\mu = \$8.9$.

$$z = \frac{8.96 - 8.9}{0.1} = 0.6$$

From the normal distribution table, for z = -0.6, the area is 0.2257.

Therefore, $\alpha = 0.5 - 0.2257 = 0.2743$.

e.



Supplementary Problems

1.

➤a.

We assume that the coin is well balanced. Therefore, H_o: $\mu = 0.5(100) = 50$.

And since a biased coin could favor either heads or tails, H_a: $\mu \ne 50$.

➤b.

Let X be the number of heads resulting from 100 tosses.

For p = 0.6, we need to find $\beta = P\{44.5 \le X \le 55.5\}$.

$$\sigma = \sqrt{100(0.6)(0.4)} \approx 4.9$$

$\mu = 0.6(100) = 60$

$$z = \frac{44.5 - 60}{4.9} = -3.16$$

$$z = \frac{55.5 - 60}{4.9} = -0.92$$

$\beta = P\{44.5 \le X \le 55.5\} = P\{-3.16 \le Z \le -0.92\}$

From the normal distribution table, $\beta = P\{-3.16 \le Z \le -0.92\} = 0.4992 - 0.3212 = 0.178$.

2.

➤a.

The claim is H_o: $\mu = 0.6(200) = 120$

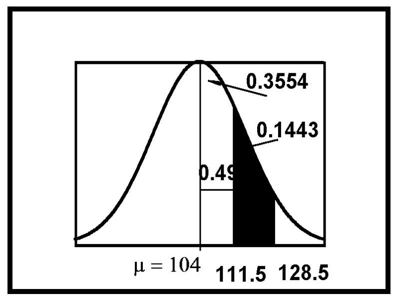
And the counter-claim is H_a: $\mu \ne 120$

➤b.

Let X be the number of students in the survey that are female.

For p = 0.52, we need to find $\beta = P\{111.5 \le X \le 128.5\}$.

b.



$$\sigma = \sqrt{200(0.52)(0.48)} \approx 7.07$$

$\mu = 0.52(200) = 104$

$$z = \frac{111.5 - 104}{7.07} = 1.06$$

$$z = \frac{128.5 - 104}{7.07} = 3.47$$

$\beta = P\{111.5 \le X \le 128.5\} = P\{1.06 \le Z \le 3.47\}$

From the normal distribution table, $\beta = P\{1.06 \le Z \le 3.47\} = 0.4997 - 0.3554 = 0.1443$.

3.

➤a.

From the claim, we set p = 0.25. The counter-claim would be p > 0.25.

$H_o: \mu = 200(0.25) = 50$

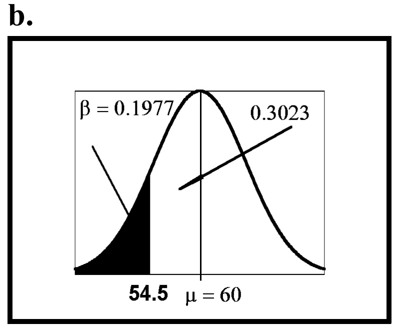
and the claim becomes the counter-claim: H_a: $\mu > 50$.

➤b.

Let X be the number of union members in the survey.

For p = 0.3, we need to find $\beta = P\{X \le 54.5\}$.

b.



$$\sigma = \sqrt{200(0.3)(0.7)} \approx 6.48$$

$\mu = 0.3(200) = 60$

$$z = \frac{54.5 - 60}{6.48} = -0.85$$

$\beta = P\{X \le 54.5\} = P\{Z \le -0.85\}$

From the normal distribution table, $\beta = P\{Z \le -0.85\} = 0.5 - 0.3023 = 0.1977$.

4.

➤a.

The claim is that 92% successfully lose significant weight loss. Therefore, $p = 0.90$ and

$H_o : \mu = 100(0.9) = 92$.

The counter-claim is H_a.: $\mu < 92$

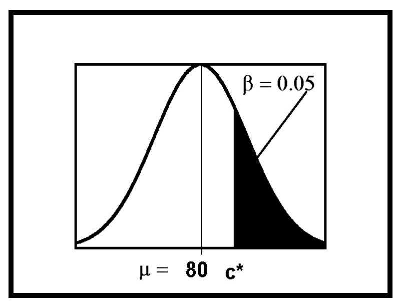
➤b.

Let X be the number of clients that lost significant weight.

D.R From the sample if $X \le c^*$ then ***reject the claim (H_0) and accept the counter-claim (H_a)****; otherwise* ***reject the counter-claim****.*

For p = 0.8, we need to find c* where $\beta = P\{X \ge c^*\} = 0.05$.

b.



$$\sigma = \sqrt{100(0.8)(0.2)} = 4$$

$\mu = 0.8(100) = 80$

$c^* = 80 + 4z$

From the normal distribution, for the area 0.5 - 0.05 = 0.45,
z = 1.64.

Therefore, $c^* = 80 + 4(1.64) = 86.56$

The decision rule would read: *From the sample if $X \le 86$ then* ***reject the claim (H_0) and accept the counter-claim (H_a)****; otherwise* ***reject the counter-claim.***

5.

➤a.

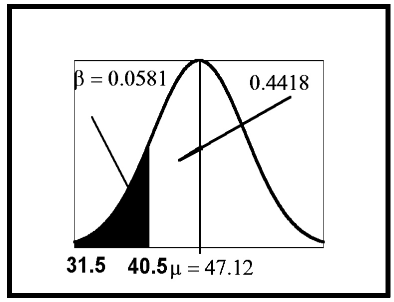
If the wheel is in balance, then p = 18/38, the probability that each spin of the wheel will result in an odd number occurring. Therefore, H_o: $\mu = 76(18/38) = 36$.

Since a bias can occur in favor or against an odd number occurring,

H_a: $\mu \neq 36$

➤b.
Let X be the number of odd numbers that occurred.

b.



For p = 0.62, we need to find $\beta = P\{31.5 \le X \le 40.5\}$.

$$\sigma = \sqrt{76(0.62)(0.38)} \approx 4.23$$

$$\mu = 0.62(76) = 47.12$$

$$z = \frac{31.5 - 47.12}{4.23} \approx -3.69$$

$$z = \frac{40.5 - 47.12}{4.23} \approx -1.57$$

From the normal distribution table, $\beta = P\{31.5 \le X \le 40.5\} = P\{-3.69 \le Z \le -1.57\} \approx$ 0.4999 - 0.4418 = 0.0581 .

6.
➤a.
Since the claim is that 35% of all viewers are males, p = 0.35. Therefore,
H_o: $\mu = 0.35(500) = 175$.

The counter-claim is that more or less than 35%. Therefore, H_a: $\mu \neq 175$.

➤b.
Let X be the number of males that watch soap operas.

For p = 0.45, we need to find $\beta = P\{169.5 \le X \le 180.5\}$.

$$\sigma = \sqrt{500(0.45)(0.55)} \approx 11.12$$

$$\mu = 500(0.45) = 225$$

$$z = \frac{180.5 - 225}{11.12} \approx -4$$

$$z = \frac{169.5 - 225}{11.12} \approx -4.99$$

From the normal distribution table,

$\beta = P\{169.5 \le X \le 180.5\} = P\{-4.99 \le Z \le -4\} \approx 0.4999 - 0.4999 = 0$

7.

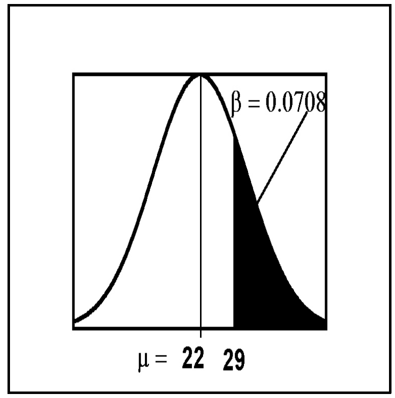
➤a.

From the decision rule it follows:

$H_o : \mu = 0.15(200) = 30$

$H_a : \mu < 30$

b.



➤b.

For p = 0.11,

$\sigma = \sqrt{200(0.11)(0.89)} \approx 4.42$

$\mu = 0.11(200) = 22$

We need to find $\beta = P\{X \ge 28.5\}$.

$$z = \frac{28.5 - 22}{4.42} \approx 1.47$$

From the normal distribution table,

$\beta = P\{X \ge 28.5\} = P\{Z \ge 1.47\} = 0.5 - 0.4292 = 0.0708.$

$P\{Z \ge 1.47\} = 0.5 - 0.4292 = 0.0708$

8.

➤a.

Since the CEO is biased in his claim, we assume $p \le 0.01$ and $p > 0.01$. Therefore,

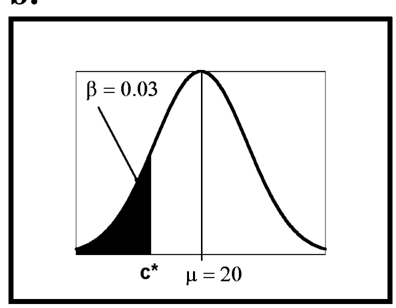
$H_o : \mu = (0.01)1000 = 10$

$H_a : \mu > 10$

➤b.

D.R. If $x \le c^*$ then reserve judgement or accept the claim that only 1% of their trucks have been recalled.

b.



p = 0.02.

$\mu = (0.02)1000 = 20$

$\sigma = \sqrt{0.02(0.98)1000} \approx 4.23$

$c^* = 20 - z\sigma$

We need to find z for the area

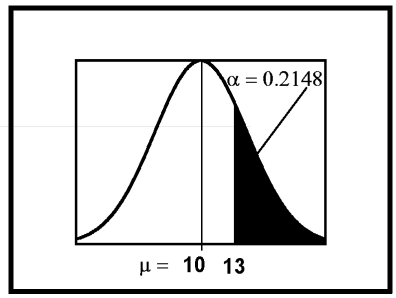
$\beta = 0.5 - 0.47 = 0.03.$

From the normal distribution table, z = 1.88.

$c^* = 20 - 4.23(1.88) \approx 12.05$

D.R. If x ≤ 12 then reserve judgement or accept the claim that only 1% of their trucks have been recalled.

c.



➤c.

$\mu = (0.01)1000 = 10$

$\sigma = \sqrt{0.01(0.91)1000} \approx 3.15$

From the decision rule we need to find $\alpha = P\{X \geq 12.5\}$.

$$z = \frac{12.5 - 10}{3.15} \approx 0.79$$

From the normal distribution table, $\alpha = P\{X \geq 12.5\} = P\{Z \geq 0.79\} = 0.5 - 0.2852 = 0.2148$.

9.

We us the formula $\mu = \overline{X} - z\frac{\sigma}{\sqrt{N}} = 16.2 - \frac{-1.64(1.5)}{\sqrt{100}} = 16.45$

Setting $\beta = 0.05$, we find from Table C, $z = -1.64$.

Therefore, since $\beta \leq 0.05$, we have $\mu \geq 16.45$

10.

We use the Bayes formula: $P(\mu = 16|\overline{X} \geq 16.2) = \frac{P(\overline{X} \geq 16.2|\mu = 16)P(\mu = 16)}{P(\overline{X} \geq 16.2)}$.

Step 1: First we will find $P(\overline{X} \geq 16.2|\mu = 16)$.

$z = (16.2 - 16)/0.15 = 1.33.$

From Table C, we find $P(\overline{X} \geq 16.2|\mu = 16) = 0.5 - 0.4082 \approx 0.092$.

Step 2: From the manufacturer's claim, we have $P[\mu = 16] = 0.90$.

Step 3: We can write $(\overline{X} \geq 16.2) = \{(\overline{X} \geq 16.2)\cap(\mu = 16)\}\cup\{(\overline{X} \geq 16.2)\cap(\mu > 16)\}$.

Step 4: $P(\overline{X} \geq 16.2) = P\{(\overline{X} \geq 16.2) \cap (\mu = 16)\} + P\{(\overline{X} \geq 16.2) \cap (\mu > 16)\}$

Step 5: $P\{(\overline{X} \geq 16.2) \cap (\mu = 16)\} = P(\mu = 16)P(\overline{X} \geq 16.2 | \mu = 16) = (0.90)(0.092) \approx 0.083$

$P\{[(\overline{X} \geq 16.2) \cap (\mu > 16)\}] = P(\mu > 16)P(\overline{X} \geq 16.2 | \mu > 16)$

$P(\overline{X} \geq 16.2 | \mu > 16) = 1 - P(\overline{X} < 16.2 | \mu > 16) = 1 - 0.05 = 0.95$

Therefore, $P\{(\overline{X} \geq 16.2) \cap (\mu > 16)\} = P(\mu > 16)P(\overline{X} \geq 16.2 | \mu > 16) = (0.10)(0.95) = 0.095$

Step 6:
$P(\overline{X} \geq 16.2) = P\{(\overline{X} \geq 16.2) \cap (\mu = 16)\} + P\{(\overline{X} \geq 16.2) \cap (\mu > 16)\} = 0.083 + 0.095 \approx 0.18$

Step 7:Finally,

$$P\{\mu = 16 | \overline{X} \geq 16.2\} = \frac{P\{\overline{X} \geq 16.2 | \mu = 16\}P(\mu = 16)}{P(\overline{X} \geq 16.2)} = 0.083/0.18 = 0.46$$

11. For a certain population, the following claim and counter-claim was made:

H_o: $\mu = 10$
H_a: $\mu > 10$

To test this claim a sample of N = 100 is to be taken. For the following decision rule:

D.R.: If $\overline{x} \geq c*$ *then* ***reject*** H_o ***and accept*** H_a; *otherwise* ***reject*** H_a.

➤a.

N = 100

$$\sigma_{\overline{X}} = \frac{2}{\sqrt{100}} = 0.2$$

c* = 10 + z(0.2)

Since $\alpha = 0.01$, we look up 0.49 from Table C for z = 2.33.

c* = 10 + z(0.2) = 10 + 2.33(0.2) = 10 + 0.466 ≈ 10.47.

D.R.: If $\overline{x} \geq 10.47$ *then* ***reject*** H_o ***and accept*** H_a; *otherwise* ***reject*** H_a.

➤b.
Find the minimal range of values of μ for $0 \le \beta \le 1$.

We use the formula: $\mu = \bar{x} - z\sigma_{\bar{X}}$

which gives $z = \dfrac{\bar{X} - \mu}{\sigma_{\bar{X}}}$ = (10.47 - 10.01)/0.2 = 2.3

For $\mu = 10.01$, $z = 2.3$.

From Table C, $P(\bar{x} \le 10.47) = 0.5 + 0.4893 \approx 0.99$.

For the lower bound we use $\mu = \bar{x} - z\sigma_{\bar{X}}$.

For $\beta = 0$, for the lower bound we use $\mu = \bar{x} - z\sigma_{\bar{X}} = 10.47 + 3.59(0.2) \approx 11.19$.

$10.01 \le \mu \le 11.19$