Probability theory
Lesson 24
The Poisson Approximation to the Binomial Distribution

24.1- Applications

24.1 - Problem 1:

➤(a).

N = 4

k = 2

p = 0.05

q = 1 - 0.05 = 0.95

$$P\{X = 2\} = \binom{N}{k} p^k q^{N-k} = \binom{4}{2}(0.05)^2 (0.95)^2 \approx 0.014$$

➤(b).

Step 1: Since we are approximating the binomial distribution with the Poisson distribution:

$\mu = Np = 4(0.05) = 0.2$

Step 2: $P\{X = 2\} = \frac{\mu^k}{k!} e^{-\mu} \approx \frac{0.2^2}{2!}(2.718)^{-0.2} = 0.016$

24.1 - Problem 2:

Step 1: Since we are approximating the binomial distribution with the Poisson distribution:

$\mu = Np = 300(0.01) = 3$

Step 2: $P\{X \geq 3\} = 1 - P\{X \leq 2\}$

Step 3: Using the Poisson table for $\mu = 3$ and $x = 2$, we have

$P\{X \leq 2\} = 0.4232$.

Therefore,

$P\{X \geq 3\} = 1 - P\{X \leq 2\} = 1 - 0.4232 = 0.5768$

Supplementary Problems

For the following problems, use the Poisson approximation to the Binomial distribution.

1.

$\mu = Np = 2000(0.001) = 2$

Let X be the random variable that equals the number of pilots that die in airplane accidents.

$P\{X > 2\} = 1 - P\{X \leq 2\} = 1 - [P\{X = 0\} + P\{X = 1\} + P\{X = 2\}] \approx$

$$1 - [\frac{2^0}{0!}(2.718)^{-2} + \frac{2^1}{1!}(2.718)^{-2} + \frac{2^2}{2!}(2.718)^{-2}] \approx 1 - 0.677 = 0.323$$

2.

$\mu = Np = 100(0.03) = 3$

Let X be the random variable that equals the number of his students that will fail.

$$P\{X = 0\} \approx \frac{3^0}{0!}(2.718)^{-3} \approx 0.0498$$

3.

➤(a).

$\mu = Np = 200(0.02) = 4$

Let X be the random variable that equals the number of students that will study overseas.

$P\{X \geq 10\} = 1 - P\{X \leq 9\} =$

$1 - [P\{X = 0\} + P\{X = 1\} + P\{X = 2\} + P\{X = 3\} + ... + P\{X = 9\}] \approx$

$$1 - [\frac{4^0}{0!}(2.718)^{-4} + \frac{4^1}{1!}(2.718)^{-4} + ... + \frac{4^9}{9!}(2.718)^{-4}] =$$

$$1 - [\frac{4^0}{0!} + \frac{4^1}{1!} + ... + \frac{4^9}{9!}](2.718)^{-4} \approx 0.0081$$

➤(b).

$$P\{X \leq 7\} \approx [\frac{4^0}{0!} + \frac{4^1}{1!} + ... + \frac{4^7}{7!}](2.718)^{-4} \approx 0.9489$$

➤(c).

$P\{X = 5\} \approx \frac{4^5}{5!}(2.718)^{-4} \approx 0.1563$

4.

➤a.

$P\{X = 10\} = \binom{100}{10}(0.1^{10})(0.90^{90}) \approx 0.13$

➤b.

$\mu = (0.1)100 = 10$

$P\{X = 10\} = \frac{\mu^k}{k!}e^{-\mu} = \frac{10^{10}}{10!}e^{-10} \approx 0.13$

5.

➤a.

$P(X \geq 2) = 1 - P(X = 0) - P(X = 1) = 1 - \binom{100}{0}(0.98^{100}) - \binom{100}{1}(0.02)(0.98)^{99} \approx 1 - 0.13 - 0.27$

$= 0.60.$

➤b.

Since

$\mu = 0.02(100) = 2,$

we have from the Poisson formula,

$P(Y \geq 2) = 1 - P(Y = 0) - P(Y = 1) = 1 - \frac{2^0}{0!}e^{-2} - \frac{2^1}{1!}e^{-2} \approx 1 - 0.14 - 0.27 = 0.59$

➤c.

Since we want in the box at least 100 non defective drives, then the minimum number in the box would be N = 100 + m where there are at most m defective drives.

Step 1: $\mu = 0.02N = 2 + 0.02m$.

Step 2: Next, we find the probability that there are at most m defective drives in the box with a probability of at least 85%:

$P(X \leq m) = P(X = 0) + P(X = 1) + ... + P(X = m) =$

$e^{-(2+0.02m)}\{\frac{(2+0.02m)^0}{0!} + \frac{(2+0.02m)^1}{1!} + ... + \frac{(2+0.02m)^m}{m!}\} \geq 0.80$

Step 3: We will select values of m = 0,1, ... to find the minimum m that satisfies the inequality.

m = 0: $P(X = 0) = e^{-2} = 0.13$.

m = 1: $P(X \leq 1) = P(X = 0) + P(X = 1) = e^{-2 - 0.02}\{1 + 2.02\} = e^{-2.02}(3.02) \approx 0.40$.

m = 2: $P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2) = e^{-2 - 0.04}\{1 + 2.04 + (2.04^2)/2\} \approx e^{-2.04}(5.12)$

≈ 0.67

m = 3: $P(X \leq 3) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) =$

$e^{-2.06}\{1 + 2.06 + (2.06^2)/2 + (2.06^3)/6\} = 0.84$

Therefore, the box should contain a minimum of 103 drives.

6.
Since there are 10 digits, p = 1/10 = 0.10.

We need to find a minimum size N.

Let X be the r.v. that equals the number of times the digit 5 appear when generated N digits.

Therefore,

$$P(X = k) = \frac{\mu^k}{k!} e^{-\mu} \approx \frac{(0.10N)^k}{k!}(e)^{-0.10N}$$

$P(X \geq 1) \geq 0.90$.

We first compute $P(X = 0) = 1 - P(X \geq 1) \leq 1 - 0.90 = 0.10$.

$$P(X = 0) = \frac{(0.10N)^0}{0!}(e)^{-0.10N} = e^{-0.10N} \leq 0.10$$

$10 \leq e^{0.10N}$

$\ln 10 \leq \ln e^{0.10N} = 0.10N$

$2.30 \leq 0.10N$

$23 \leq N$

7.

➤a

Assume $0 \leq k \leq r$. The Hypergeometric distribution is

$$P(X = k) = \frac{\binom{n}{k}\binom{N-n}{r-k}}{\binom{N}{r}}$$

$P(X = 0) + P(X = 1) + \ldots + P(X = k) =$

$$\frac{\binom{n}{0}\binom{N-n}{r-0} + \binom{n}{1}\binom{N-n}{r-1} + \ldots + \binom{n}{r}\binom{N-n}{r-r}}{\binom{N}{r}} = 1$$

$$\binom{n}{0}\binom{N-n}{r-0} + \binom{n}{1}\binom{N-n}{r-1} + \ldots + \binom{n}{r}\binom{N-n}{r-r} = \binom{N}{r}$$

Let N - n = m. Substituting N - n above gives

$$\binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \ldots + \binom{n}{r}\binom{m}{0} = \binom{n+m}{r}$$

➤b.

Step 1: First we have

$$\binom{N}{k} = \binom{N}{N-k}$$

Step 2: From a. we have

$$\binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \ldots + \binom{n}{r}\binom{m}{0} = \binom{n+m}{r}$$

Let n = r = N.

Substituting above we have

$$\binom{N}{0}\binom{N}{N} + \binom{N}{1}\binom{N}{N-1} + \dots + \binom{N}{N}\binom{N}{0} = \binom{N+N}{N}$$

From Step 1: we have

$$\binom{N}{0}\binom{N}{0} + \binom{N}{1}\binom{N}{1} + \dots + \binom{N}{N}\binom{N}{N} = \binom{2N}{N}$$

$$\binom{N}{0}^2 + \binom{N}{1}^2 + \dots + \binom{N}{N}^2 = \binom{2N}{N}$$

8.
➤a.

X: The r.v. equal to the number of heads tossed by Jack for N tosses.

Y: The r.v. equal to the number of heads tossed by Jill for N tosses

E: The event that the number of heads tossed by Jack and Jill are equal.

Step 1:

$\mathbf{E} = [(X = 0)\cap(Y = 0)]\cup[(X = 1)\cap(Y = 1)]\cup[(X = 2)\cap(Y = 2)]\cup \dots \cup[(X = N)\cap(Y = N)]$

$P(\mathbf{E}) = P[(X = 0)P(Y = 0)]+P[(X = 1)P(Y = 1)]+P[(X = 2)P(Y = 2)]+ \dots +P[(X = N)P(Y = N)] =$

$P(X = 0)^2+ P(X = 1)^2+ P(X = 2)^2 + \dots + P(X = N)^2 =$

$$[\binom{N}{0}(.5)^N]^2 + [\binom{N}{1}(0.5)^{N-1}(0.5)^1]^2 + [\binom{N}{2}(0.5)^{N-2}(0.5)^2]^2 + \dots + =$$

$$[\binom{N}{N}(0.5)^0(0.5)^N]^2$$

$$[\binom{N}{0}^2 + \binom{N}{1}^2 + \binom{N}{2}^2 + \dots + \binom{N}{N}^2](0.5^{2N}) = \binom{2N}{N}(0.5)^{2N}$$

➤b.

Let $P(X = k) = P(Y = k) = \frac{\mu^k}{k!} e^{-\mu} \approx \frac{(0.5N)^k}{k!}(e)^{-0.5N}$

$P(\mathbf{E}) = P(X = 0)^2 + P(X = 1)^2 + P(X = 2)^2 + ... + P(X = N)^2 =$

$$[\frac{(0.5N)^0}{0!}(e)^{-0.5N}]^2 + [\frac{(0.5N)^1}{1!}(e)^{-0.5N}]^2 + ... + [\frac{(0.5N)^N}{N!}(e)^{-0.5N}]^2 =$$

$$[\frac{(0.5N)^0}{0!^2} + \frac{(0.5N)^2}{1!^2} + ... + \frac{(0.5N)^{2N}}{N!^2}](e)^{-N}$$

9.

Step 1: For large N and small p, we know that the Poisson distribution approximates the binomial distribution:

$$\frac{\mu^k}{k!} e^{-\mu} \approx \binom{N}{k}(p)^k q^{N-k}, \text{ where } q = 1 - p.$$

Step 2: Since $\mu = pN$, we can write

$p = \mu/N$.

Since the Poisson distribution approximates the binomial distribution, we should expect σ^2 of the Poisson distribution to approximate the variance of the binomial distribution

$Npq = \mu q$.

As N gets arbitrary large, p will approach zero and therefore

$q = 1 - p \approx 1$.

Therefore, it is reasonable to assume

$\sigma^2 = \mu$.