Probability theory
Lesson 22
The Poisson Distribution

22.1- The Poisson Time Distribution.

22.1 - Problem 1:
Step 1: $\mu_t = \mu t = 3$, the average of 3 students at a local university violate curfew restrictions where $t = 2$.

Step 2: $k = 1$, the number of students that violate the curfew restrictions.

Step 3: $P\{X = 1\} = \dfrac{(\mu t)^k}{k!} e^{-\mu t} \approx \dfrac{3^1}{1!} 2.718^{-3} \approx 0.15$.

22.1 - Problem 2:
Step 1: $k = 5$, since we are interested in the probability that exactly 5 customers request an oil change.

Step 2: The time period we are concerned with is $t = 3$ hours. Therefore, we need to compute μ .

From the statement of the problem, we know that $2\mu = 3$. Therefore,

$\mu = 3/2$, average number of oil change requests per hour .

Since $t = 3$, we have $t\mu = 3(1.5) = 4.5$, average number of oil change requests over 3 hours.

Step 3: $P\{X = 5\} = \dfrac{(\mu t)^k}{k!} e^{-\mu t} \approx \dfrac{(4.5)^5}{5!} 2.718^{-4.5} \approx 0.17$ the probability that 5 customers request an oil change over 3 hours.

22.1 - Problem 3:
➤(a).
Step 1: Compute μ.

Since $\mu_t = 12\mu = 6$, then $\mu = 0.5$ average number of fires per month.

Step 2: Compute the average number of fires over 2 months gives

$\mu_t = \mu t = 2\mu = 2(0.5) = 1$, the average number of fires over 2 months.

Step 3: To compute the probability that at least 1 fire occurs over 2 months, we use the formula

$P\{X \geq 1\} = 1 - P\{X \leq 0\} = 1 - P\{X = 0\}$.

Therefore, we need to use the Poisson formula for k = 0.

For k = 0, and t = 2 months, we have

$$P\{X = 0\} = \frac{(\mu t)^k}{k!}e^{-\mu t} \approx \frac{(1)^0}{0!}2.718^{-1} \approx 0.362.$$

➤(b).
The probability that at most 1 fire can occur over 2 months can be written

$P\{X \leq 1\} = P\{X = 0\} + P\{X = 1\}$.

From (a), we know that

$\mu_t = \mu t = 2\mu = 2(0.5) = 1$,

the average number of fires over 2 months.

Since we need to compute

$P\{X = 0\}$

and

$P\{X = 1\}$,

we have

$$P\{X = 0\} = \frac{(\mu t)^k}{k!}e^{-\mu t} \approx \frac{(1)^0}{0!}2.718^{-1} \approx 0.36$$

$$P\{X = 1\} = \frac{(\mu t)^k}{k!}e^{-\mu t} \approx \frac{(1)^1}{1!}2.718^{-1} \approx 0.36$$

Therefore,

$P\{X \leq 1\} = P\{X = 0\} + P\{X = 1\} \approx 0.36 + 0.36 = 0.72$.

22.1 - Problem 4:
Step 1: The average number of orders returned over t = 7 days is

$\mu_t = \mu t = (3\%)2000 = (0.03)2000 = 60$.

Step 3: Computing the average number of orders returned over 1 day gives

$\mu_t = \mu t = 60$
$t = 7$
$\mu_t = \mu t = \mu 7 = 60$

$\mu = 60/7$, average number of orders returned over 1 day.

Here t = 1.

Step 5: The number is orders returned is k = 5.

Step 6: $P\{X = 5\} = \frac{(\mu t)^k}{k!} e^{-\mu t} \approx \frac{(\frac{60}{7})^5}{5!} 2.718^{-\frac{60}{7}} \approx 0.073.$

22.2 - The Poisson Spatial Distribution.

22.1 - Problem 1:
Step 1: Let t = 10, the number of pages.

Step 2: $\mu = 0.25$

Step 3: The average number of errors over 10 pages is

$\mu_t = \mu t = 0.25(10) = 2.5.$

Step 4: The number of errors is

k = 2.

Step 5: The probability that only 2 errors occur over 10 pages is

$P\{X = k\} = \frac{(\mu t)^k}{k!} e^{-\mu t} = P\{X = 2\} \approx \frac{(2.5)^2}{2!} 2.718^{-2.5} \approx 0.26.$

22.2 - Problem 2:
Step 1: The average number of coaxial cable found defective, for every 100,000 feet stalled $100{,}000\mu = 10$.

Step 2: Computing μ, $\mu = \dfrac{10}{100000} = 0.0001$, average per foot.

Step 3: The average number of coaxial cable found defective, for every 10,000 feet stalled is

t = 10,000

$\mu_t = (10{,}000)0.0001 = 1.$

Step 4: The probability that 2 feet are defective is

$$P\{X = k\} = \frac{(\mu t)^k}{k!}e^{-\mu t} = P\{X = 2\} \approx \frac{1^2}{2!}2.718^{-2} \approx 0.07.$$

22.2 - Problem 3:

➤(a).

Step 1: The average number of clients that like the computer is

$\mu_t = \mu t = 35\%(1{,}000) = (0.35)(1000) = 350$ and

$\mu = 0.35$

Step 2: The average number of clients out of the 10 sent that like the computer is

$\mu_t = 10\mu = 10(0.35) = 3.5.$

Step 3: The probability of not continuing sales on the computer is

$$P\{X \le 1\} = P\{X = 0\} + P\{X = 1\} \approx \frac{3.5^0}{0!}2.718^{-3.5} + \frac{3.5^1}{1!}2.718^{-3.5} \approx 0.136$$

➤(b).

Step 1: The average number of clients that like the computer is

$\mu_t = \mu t = 20\%(1{,}000) = (0.20)(1000) = 200$ and

$\mu = 0.20$

Step 2: The average number of clients out of the 10 sent that like the computer is

$\mu_t = 10\mu = 10(0.20) = 2$

Step 3: The probability of continuing sales on the computer is

$P\{X \geq 2\} = 1 - P\{X = 0\} - P\{X = 1\} \approx 1 - \frac{2^0}{0!}2.718^{-2} - \frac{2^1}{1!}2.718^{-2} \approx 0.594.$

Supplementary Problems

1.
Step 1: t = 100, and $\mu_t = t\mu = 3$, average per box.

Step 2: $P\{X = k\} = p = \frac{(\mu t)^k}{k!}e^{-\mu t} = P\{X = 1\} \approx \frac{3^1}{1!}2.718^{-3} \approx 0.149,$

the probability that a single box contains exactly one broken bulb.

Step 3: Applying the binomial distribution for

N = 5,
k = 3
p = 0.149

we have

$P\{Y = 3\} = \binom{5}{3}(0.149)^3(0.851)^2 \approx 10(0.0033)(0.724) \approx 0.024.$

2.
➤a.
Step 1: **E:** The event that the box contains 1 defective transmission.
W: The event that the transmission came from the West coast.
W′: The event that the transmission came from the East coast.
X: The random variable of the number of defective parts in a box.

$P(\mathbf{W}') = \frac{10{,}000}{25{,}000} = 0.40$

$P(\mathbf{W}) = 0.60$

Step 2: $\mathbf{E} = (\mathbf{E}\cap\mathbf{W})\cup(\mathbf{E}\cap\mathbf{W}')$

$P(\mathbf{E}) = P(\mathbf{E}\cap\mathbf{W}) + P(\mathbf{E}\cap\mathbf{W}')$

$P(\mathbf{E}\cap\mathbf{W}) = P(\mathbf{E}|\mathbf{W})P(\mathbf{W}) = P(\mathbf{X} = 1|\mathbf{W})P(\mathbf{W}) = P(\mathbf{X} = 1|\mathbf{W})(0.60)$

$P(\mathbf{E} \cap \mathbf{W}') = P(\mathbf{E}|\mathbf{W}')P(\mathbf{W}') = P(\mathbf{X} = 1|\mathbf{W}')P(\mathbf{W}') = P(\mathbf{X} = 1|\mathbf{W}')(0.40)$

Step 3: For the East coast the average number of defective parts is

$\mu = 500/10{,}000 = 0.05$.

Therefore, the average number of defective transmission per box from the East coast is

$\mu_t = \mu t = 0.05(100) = 5$

For 1 defective part we have k = 1 and therefore,

$$P\{X = k|\mathbf{W}'\} = \frac{(\mu t)^k}{k!}e^{-\mu t} = P\{X = 1|\mathbf{W}'\} \approx \frac{5^1}{1!}2.718^{-5} \approx 0.03,$$

the probability that the box contains one defective transmission.

Step 4: For the West coast the average number of defective parts is

$\mu = 450/15{,}000 = 0.03$.

Therefore, the average number of defective transmission per box from the West coast is

$\mu_t = \mu t = 0.03(100) = 3$.

For 1 defective part, we have k = 1 and therefore

$$P\{X = k|\mathbf{W}\} = \frac{(\mu t)^k}{k!}e^{-\mu t} = P\{X = 1|\mathbf{W}\} \approx \frac{3^1}{1!}2.718^{-3} \approx 0.15,$$

the probability that the box contains exactly 1 defective transmission.

Step 5: $P(\mathbf{E}) = P(\mathbf{E} \cap \mathbf{W}) + P(\mathbf{E} \cap \mathbf{W}') = P(\mathbf{X} = 1|\mathbf{W})(0.60) + P(\mathbf{X} = 1|\mathbf{W}')(0.40) =$

$(0.15)(0.60) + (0.03)(0.40) = 0$

$\mathbf{D} = (\mathbf{D} \cap \mathbf{E}) \cup (\mathbf{D} \cap \mathbf{W})$

$P(\mathbf{D}) = P(\mathbf{D}\cap\mathbf{E}) + P(\mathbf{D}\cap\mathbf{W}) = P(\mathbf{E})P(\mathbf{D}|\mathbf{E}) + P(\mathbf{W})P(\mathbf{D}|\mathbf{W}) = (0.40)(0.03) + (0.60)(0.15) = 0.102$

➤b.

This is a Bayesian problem:

$$P(\mathbf{W}'|\mathbf{E}) = \frac{P(\mathbf{E}|\mathbf{W}')P(\mathbf{W}')}{P(\mathbf{E})} \approx \frac{(0.03)(0.4)}{0.10} = 0.12$$

3.

Using the Poisson distribution ,

$$P\{X = k\} = \frac{(\mu t)^k}{k!}e^{-\mu t},$$

we compute the following table:

k	$\mathbf{P\{X = k\}} = \frac{(\mu t)^k}{k!}e^{-\mu t}$
0	0.01
1	0.045
2	0.10
3	0.15
4	0.17
5	0.15

Therefore, k = 4.

4.

➤a.

Step 1: t = 1 page

μ = 1per page ,

$\mu_t = t\mu = 1$

Step 2: $P\{X \geq 3\} = 1 - P\{X \leq 2\} = 1 - [P\{X= 0\} + P\{X = 1\} + P\{X = 2\}]$

$$P\{X = k\} = \frac{(\mu t)^k}{k!} e^{-\mu t} = \frac{e^{-1}}{k!}$$

$$P\{X = 0\} = \frac{e^{-1}}{0!} = e^{-1} \approx 0.37$$

$$P\{X = 1\} = \frac{e^{-1}}{1!} = e^{-1} \approx 0.37$$

$$P\{X = 2\} = \frac{e^{-1}}{2!} = 0.37/2 \approx 0.185$$

Step 3: $P\{X \geq 3\} \geq 1 - 0.37 - 0.37 - 0.185 \approx 0.08$

➤b.
Step 1:
t = 3 pages
$\mu = 1$ per page ,
$\mu_t = t\mu = 3$

$$P\{X = 0\} = \frac{(\mu t)^k}{k!} e^{-\mu t} = \frac{3^0 e^{-3}}{0!} = e^{-3} \approx 0.05$$

5.
The basic unit is 1 person.

$\mu = 0.01$, average per person.

t = N, the sample size

$\mu_t = \mu t = (0.01)N$

$P\{X \geq 1\} = 1 - P\{X = 0\}$

$$P\{X = 0\} = \frac{(0.01N)^0}{0!} e^{-0.01N} = e^{-0.01N}$$

$P\{X \geq 1\} = 1 - e^{-0.01N} \geq 0.95$

$1 - 0.95 \geq e^{-0.01N}$

$0.05 \geq e^{-0.01N}$

$\ln(0.05) \geq \ln(e^{-0.01N}) = -0.01N$

$\ln(.05) \approx -3$

$-3 \geq -0.01N$

$300 \leq N$

6.
X: The random variable that equals the number of sales.

$P\{X \geq 1\} \geq 0.99$

$P\{X \geq 1\} = 1 - P\{X = 0\} \geq 0.99$

$1 - 0.99 \geq P\{X= 0\}$

$0.01 \geq P\{X = 0\} = \dfrac{(\mu t)^0}{0!} e^{-\mu t} = e^{-\mu t}$

$e^{\mu t} \geq 1/0.01 = 100$

$t = 1$

$e^{\mu} \geq 100$

$\ln(e^{\mu}) \geq \ln(100) \approx 5$

$\ln(e^{\mu}) = \mu \approx 5$

7.
➤a.
Step 1: First we need to find the probability p of a hand having a full house.

$\#\mathbf{S} = \binom{52}{5} = 2{,}598{,}960$

E: A full house occurs.

$$\#\mathbf{E} = 13\binom{4}{3}12\binom{4}{2} = 3744$$

$$P(\mathbf{E}) = p = \frac{3744}{2598960} = \frac{6}{4165}$$

Step 2:t = 10

$\mu = p = \frac{6}{4165}$, per hand

$$\mu t = 10(\frac{6}{4165}) = \frac{12}{833}$$

$$P\{X \geq 1\} = 1 - P\{X = 0\} = 1 - \frac{(\frac{12}{833})^0}{0!}e^{-\frac{12}{833}} = 1 - e^{-\frac{12}{833}} \approx 1 - 0.99 = 0.01$$

➤b.

N: Number of hands

$$\mu N = (\frac{6}{4165})N$$

$$P\{X \geq 1\} > 1/2$$

$$P\{X = 0\} = \frac{(\frac{6}{4165}N)^0}{0!}e^{-\frac{6}{4165}N} = e^{-\frac{6}{4165}N}$$

$$P\{X \geq 1\} = 1 - P\{X = 0\} = 1 - e^{-\frac{6}{4165}N} > 1/2$$

$$e^{-\frac{6}{4165}N} < 1/2$$

$$2 < e^{\frac{6}{4165}N}$$

$$\ln(2) < N(6/4165)$$

(0.69)(4165/6) < N

478 < N

8.

➤a.

We need to use the formula, from supplementary 16, lesson 17 where N = 5,

$$\binom{N}{2} ({}_{365}P_{N-1})\, 365^{-N}$$

is the probability that for N random people, 2 are born on the same day and the others on different days.

For our problem, let N = 10. Therefore, the probability that for a given group,

$$p = \binom{10}{2} ({}_{365}P_9)\, 365^{-10} = 45\ {}_{365}P_9 365^{-10} \approx 45(0.002) = 0.09.$$

Now we apply the Poisson distribution:

$t = 100$
$\mu = 0.09$
$\mu t = 9$

$k = 10\%(100) = 0.10(100) = 10$

$$P\{X = 10\} = \frac{(\mu t)^k}{k!} e^{-\mu t} = \frac{9^{10} e^{-9}}{10!} \approx 0.12.$$

➤b.

We need to use the formula, from supplementary 16, lesson 17 where N = 5,

$$\binom{N}{2} ({}_{365}P_{N-1})\, 365^{-N}$$

is the probability that for N random people, 2 are born on the same day and the others on different days.

For our problem, let N = 5. Therefore, the probability that for a given group,

$$p = \binom{5}{2} ({}_{365}P_4)\, 365^{-5} = 10\, {}_{365}P_4 365^{-5} \approx 10(0.003) = 0.03.$$

Now we apply the Poisson distribution:

$t = 200$

$\mu = 0.03$

$\mu t = 6$

$k = 10\%(200) = 0.10(200) = 20$

$$P\{X = 20\} = \frac{(\mu t)^k}{k!} e^{-\mu t} = \frac{6^{20} e^{-6}}{20!} \approx 0.$$

9.

Step 1:

E: The event that at least 1 page of the book contains more than k errors.

E′: The event that each page of the book contains at most k errors.

$\mathbf{A}_j$: The event that the jth page of the book contains at most k errors, where each of these pages are independent.

Step 2: Let X be a random variable with a Poisson distribution such that

$$P(X = v) = \frac{(\lambda t)^v}{v!} e^{-\lambda t}\ .$$

Since t = 1 represents v errors per page, we have

$$P(X = v) = \frac{(\lambda)^v}{v!} e^{-\lambda} = p(v,\lambda);\ v = 0,1,2,....$$

Therefore,

$$P(\mathbf{A}_j) = P(X \le k) = P(X = 0) + P(X = 1) + ... + P(X = k) = p(0,\lambda) + p(1,\lambda) + ... + p(k,\lambda) = p.$$

$\mathbf{E}' = \mathbf{A}_1 \cap \mathbf{A}_2 \cap \mathbf{A}_3 \cap \ldots \cap \mathbf{A}_n$

From independence, $P(\mathbf{E}') = P(\mathbf{A}_1 \cap \mathbf{A}_2 \cap \mathbf{A}_3 \cap \ldots \cap \mathbf{A}_n) = P(\mathbf{A}_1)P(\mathbf{A}_2)\mathbf{P}(\mathbf{A}_3)\ldots P(\mathbf{A}_n) = p^n$,

$P(\mathbf{E}) = 1 - p^n$.

10.

➤a.

To compute μ, we complete the table below:

k:Number of false alarms	0	1	2	3	4	5
H_k:Number of hours where k false alarms occurred	29	27	11	4	1	0
kH_k:	0	27	22	12	4	0

$\mu = (0 + 27 + 22 + 12 + 4 + 0)/72 \approx 0.903$

➤b.

k:Number of false alarms	0	1	2	3	4	5
$H_k = (72)\dfrac{(0.903)^k}{k!}e^{-0.903}$ **Number of hours where k false alarms occurred**	29.18	26.35	11.90	3.58	1	0

11.

➤a.

To compute μ, we complete the table below:

k:Number of infections	0	1	2	3	4	6	7	8	9
A_k: Number of areas where k infections exists.	200	112	80	51	27	15	4	6	5
kA_k:	0	112	160	153	108	90	28	48	45

$\mu = (0 + 112 + 160 + 153 + 108 + 90 + 28 + 48 + 45)/500 \approx 1.49$

➤b.

k:Number of infections	0	1	2	3	4	5	6	7	8	9
A_k = $(500)\frac{(1.49)^k}{k!}e^{-1.49}$ **Number of areas where k infections exists**	112.69	167.9	125.09	62.13	23.14	6.9	1.71	0	0	0

12.
$P(X = k) = [(\mu t)^k e^{-\mu t}]/k!$

For this problem $\mu = 3$, $t = 2$, per 100,000 residents.

Therefore,

$P(X = k) = [(6)^k e^{-6}]/k!$

➤a.
k = 8

$P(X = 8) = [(6)^8 e^{-6}]/8! \approx 0.1033$

➤b.
$P(4 \leq X \leq 8) = P(X = 4) + P(X = 5) + P(X = 6) + P(X = 7) + P(X = 8) =$

$[(6)^4 e^{-6}]/4! + [(6)^5 e^{-6}]/5! + [(6)^6 e^{-6}]/6! + [(6)^7 e^{-6}]/7! + [(6)^8 e^{-6}]/8! \approx 0.696$

➤c.
$P(X < 3) = P(X = 0) + P(X = 1) + P(X = 2) = [(6)^0 e^{-6}]/0! + [(6)^1 e^{-6}]/1! + [(6)^2 e^{-6}]/2! \approx 0.062$

➤d.
$P(X > 3) = 1 - [P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)] =$

$1 - [(6)^0 e^{-6}]/0! + [(6)^1 e^{-6}]/1! + [(6)^2 e^{-6}]/2! + [(6)^3 e^{-6}]/3!] \approx 1 - 0.151 = 0.849$

13.
$P(X = k) = [(\mu t)^k e^{-\mu t}]/k!$

For this problem $\mu = 2.5$, $t = 1$, per minute. Therefore,

$P(X = k) = [(2.5)^k e^{-2.5}]/k!$.

➤a.

$P(X = 3) = [(2.5)^3 e^{-2.5}]/3! \approx 0.2138$

➤b.

$P(X < 5) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) =$

$(2.5)^0 e^{-2.5}/0! + (2.5)^1 e^{-2.5}/1! + (2.5)^2 e^{-2.5}/2! + (2.5)^3 e^{-2.5}/3! + (2.5)^4 e^{-2.5}/4! \approx 0.8912$

➤c.

$P(X > 6) = 1 - P(X \le 6) =$

$1 - \{P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) + P(X = 6)\} =$

$1-\{(2.5)^0 e^{-2.5}/0!+(2.5)^1 e^{-2.5}/1!+(2.5)^2 e^{-2.5}/2!+(2.5)^3 e^{-2.5}/3!+(2.5)^4 e^{-2.5}/4!+(2.5)^5 e^{-2.5}/5!+(2.5)^6 e^{-2.5}/6! \approx$

0.0142

14.

➤a.

$P(X \ge 2|X \le 4) = P[(X \le 4)\cap(X \ge 2)]/P(X \le 4) = P(2 \le X \le 4)/P(X \le 4)$

$P(2 \le X \le 4) = P(X = 2) + P(X = 3) + P(X = 4) = 2^2 e^{-2}/2! + 2^3 e^{-2}/3! + 2^4 e^{-2}/4! \approx 0.271 + 0.180 + 0.09$

$= 0.541$

$P(0 \le X \le 4) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) \approx$

$0.135+0.271+0.271 + 0.180 + 0.09 \approx 0.947$

$P(X \ge 2|X \le 4) = 0.541/0.947 \approx 0.571$

➤b.

$P(X \le 4|X \ge 2) = P[(X \le 4)\cap(X \ge 2)]/P(X \ge 2) = P(2 \le X \le 4)/P(X \ge 2)$

$P(X \ge 2) = 1 - P(X \le 1) \approx 1 - 0.406 = 0.594$

15.

➤a. We will use the binomial expansion: $(\lambda + \mu)^m =$

$$\binom{m}{0}\lambda^m\mu^0 + \binom{m}{1}\lambda^{m-1}\mu^1 + \binom{m}{r}\lambda^{m-r}\mu^r + \dots + \binom{m}{m}\lambda^0\mu^m$$

where $\binom{m}{r} = \dfrac{m!}{r!\,(m-r)!}$

Z = X + Y

P(Z=m)= P(X=m)P(Y=0)+P(X = m-1)P(Y = 1) +...+ P(X = m - r)P(Y = r)+ ..+ P(X= 0)P(Y = m)

$$= \frac{\lambda^m\mu^0}{m!0!}e^{-\lambda}e^{-\mu} + \frac{\lambda^{m-1}\mu^1}{(m-1)!1!}e^{-\lambda}e^{-\mu} + \ldots + \frac{\lambda^{m-r}\mu^r}{(m-r)!r!}e^{-\lambda}e^{-\mu} + \ldots + \frac{\lambda^0\mu^m}{(0)!m!}e^{-\lambda}e^{-\mu} =$$

$$e^{-\lambda}e^{-\mu}\left[\frac{\lambda^m\mu^0}{m!0!} + \frac{\lambda^{m-1}\mu^1}{(m-1)!1!} + \ldots + \frac{\lambda^{m-r}\mu^r}{(m-r)!r!} + \ldots + \frac{\lambda^0\mu^m}{(0)!m!}\right] =$$

$$\frac{e^{-(\lambda+\mu)}}{m!}\left[\frac{m!\lambda^m\mu^0}{m!0!} + \frac{m!\lambda^{m-1}\mu^1}{(m-1)!1!} + \ldots + \frac{m!\lambda^{m-r}\mu^r}{(m-r)!r!} + \ldots + \frac{m!\lambda^0\mu^m}{(0)!m!}\right] =$$

$$\frac{e^{-(\lambda+\mu)}}{m!}\left[\binom{m}{0}\lambda^m\mu^0 + \binom{m}{1}\lambda^{m-1}\mu^1 + \ldots + \binom{m}{r}\lambda^{m-r}\mu^r + \ldots + \binom{m}{m}\lambda^0\mu^m\right] =$$

$$\frac{e^{-(\lambda+\mu)}}{m!}(\lambda+\mu)^m = P(Z = m)$$

➤b. P(X = k|X + Y = m) = P[(X = k)∩(X + Y = m)]/P(X + Y = m)

P[(X = k)∩(X + Y = m)] = P[(X = k)∩(Y = m - k)] = P(X = k)P(Y = m - k) =

$$\frac{e^{-\lambda}}{k!}\lambda^k\frac{e^{-\mu}}{(m-k)!}\mu^{m-k}$$

$$P(X + Y = m) = \frac{e^{-(\lambda+\mu)}}{m!}(\lambda+\mu)^m$$

$$P(X = k|X + Y = m) = \frac{\dfrac{e^{-(\lambda+\mu)}}{k!(m-k)!}\lambda^k\mu^{m-k}}{\dfrac{e^{-(\lambda+\mu)}(\lambda+\mu)^m}{m!}} = \frac{m!\lambda^k\mu^{m-k}}{k!(m-k)!(\lambda+\mu)^m} = \frac{\binom{m}{k}\lambda^k\mu^{m-k}}{(\lambda+\mu)^m} =$$

$$\frac{\binom{m}{k}\lambda^k\mu^{m-k}}{(\lambda+\mu)^k(\lambda+\mu)^{m-k}} = \binom{m}{k}\left(\frac{\lambda}{\lambda+\mu}\right)^k\left(\frac{\mu}{\lambda+\mu}\right)^{m-k}$$

16.

➤a.

The Poisson distribution for robot A is $P(X = k) = \frac{e^{-0.23}}{k!}(0.23)^k$

The Poisson distribution for robot B is $P(Y = k) = \frac{e^{-0.28}}{k!}(0.28)^k$

From Problem 15a, we have the Poisson distribution for the total number of defects detected by both robots per automobile is

$$P(X + Y = m) = \frac{e^{-0.23+0.28}}{m!}(0.23 + 0.28)^m = \frac{e^{-0.51}}{m!}(0.51)^m$$

Step 1: First we complete row 2 to 3 places of accuracy :

Number of k defects	0	1	2	3	4	5
P(X + Y) = k	0.6	0.306	0.078	0.013	0.002	0
Number of autos with k defects						

Step 2: The total number of automobiles assembled is 575.

To compute the third row, we multiply 575 by the probabilities in the second row:

Number of k defects	0	1	2	3	4	5
P(X + Y) = k	0.6	0.306	0.078	0.013	0.002	0
Number of autos with k defects	345	176	45	8	1	0

➤b.

From Problem 15b, we have

$$P(X = k|X + Y = m) = \binom{m}{k}\left(\frac{\lambda}{\lambda+\mu}\right)^k\left(\frac{\mu}{\lambda+\mu}\right)^{m-k}$$

where

$\lambda = 0.23$,
$\mu = 0.28$,
$m = 3, k = 1$

$$P(X = 1|X + Y = 3) = \binom{3}{1}\left(\frac{0.23}{0.51}\right)^1\left(\frac{0.28}{0.51}\right)^2 \approx 0.41 .$$

➤c. By Bayes theorem we have:

$P(X + Y = m| X = k) = P(X = k|X + Y = m)P(X + Y = m)/P(X = k) =$

$P(X + Y = 3| X = 1) = P(X = 1|X + Y = 3)P(X + Y = 3)/P(X = 1)$.

From above we have:

$P(X = 1|X + Y = 3) \approx 0.41$, $P(X + Y = 3) = 0.013$ and

$$P(X = 1) = \frac{e^{-0.23}}{1!}(0.23)^1 \approx 0.18$$

Which gives us

$P(X + Y = 3| X = 1) \approx (0.41)(0.013)/0.18 = 0.03$.

The exponential distribution. A random variable X has a exponential distribution if

$P(X \le x) = 1 - e^{-\lambda x}$ for $x \ge 0$ and

$P(X < x) = 0$ for $x < 0$.

17.
Show how the exponential distribution is related to the Poisson distribution. Also give an interpretation of the meaning of the exponential distribution.

The exponential distribution is

$$P(Y = k) = \frac{(\mu t)^k}{k!}e^{-\mu t}$$

Let $k = 0$.

This gives

$P(Y = 0) = e^{-\mu t}$

Let $\mu = \lambda$.

$P(Y = 0) = e^{-\lambda t}$

For $x \geq 0$, $P(X \leq x) = 1 - e^{-\lambda x} = 1 - P(Y = 0)$.

Assume t = x where t is time. This gives us

for $t \geq 0$, $P(X \leq t) = 1 - e^{-\lambda x} = 1 - P(Y = 0)$.

Assume we have an event that occurs under the Poisson distribution. The event (Y = 0) means that up to the time t, the event has not yet occurred. Since

$P(X \leq t) = 1 - e^{-\lambda x} = 1 - P(Y = 0)$,

$(X \leq t)$

means that the by the time t occurs the event has occurred at least once.

Therefore, $P(X \leq t)$ is the probability that an event has occurred at least once.

18.

➤ a.

$P(X > u + v|X > u) = P[(X > u + v)\cap(X > u)]/P(X > u) = P(X > u + v)/P(X > u)$

$P(X > u + v) = e^{-\lambda(u + v)}$

$P(X > u) = e^{-\lambda u}$

$P(X > u + v|X > u) = P(X > u + v)/P(X > u) = e^{-\lambda(u + v)}/e^{-\lambda u} = e^{-\lambda v} = P(X > v)$

➤ b.

This means that the random variable X ′s outcome is independent of the time that it starts.

This means we can write

$P(X > u + v - u \mid X > u - u) = P(X > v \mid X > 0) = P(X > v)$

19. $P(X > u + v|X > u) = P(X > u + v)/P(X > u) = P(X > v)$

$P(X > u + v) = P(X > u)P(X > v)$

Let $P(X > 1) = a$

$P(X > 2) = P(X > 1 + 1) = P(X > 1)P(X > 1) = a^2$

$P(X > 3) = P(X > 2 + 1) = P(X > 2)P(X > 1) = a^2 a$

Continuing by induction:

$P(X > n) = P(X > n - 1 + 1) = P(X > n - 1)P(X > 1) = a^{n-1}a = a^n$

$P(X > 1) = P(X > n/n) = P(X > 1/n + 1/n + ... + 1/n) = [P(X > 1/n)]^n = a$

$P(X > 1/n) = a^{1/n}$

$P(X > m/n) = P(X > 1/n + 1/n + ... + 1/n) = [P(X > 1/n)]^m = (a^{1/n})^m = a^{m/n} = (1/a)^{-m/n}$

Let $e^{\lambda} = 1/a$

$\ln(e^{\lambda}) = \ln(1/a) = \lambda$

Since $1/a > 1$, we have $\lambda > 0$.

Therefore, $P(X > m/n) = (1/a)^{-m/n} = (e^{\lambda})^{-m/n} = e^{-\lambda m/n}$

Now since the function

$y = e^{-\lambda x}$

is continuous for a values of x, we can define

$P(X > x) = e^{-\lambda x}$

for all irrational numbers x.

Finally $P(X \leq x) = 1 - P(X > x) = 1 - e^{-\lambda x}$