Probability theory
Lesson 16
Variance Of A Random Variable

16.1- What is the Variance of a Random Variable?

16.1 - Problem 1:
Let X equal the number of tosses.
Let $\mathbf{F}_k$ equal the event that five occurs on the k th toss.

Step 1: $P\{X = 2\} = P\{\mathbf{F}_1 \cap \mathbf{F}_2\} = (1/6)(1/6) = 1/36$

Step 2: $P\{X = 3\} = P\{(\mathbf{F}_1 \cap \mathbf{F}_2' \cap \mathbf{F}_3) \cup (\mathbf{F}_1' \cap \mathbf{F}_2 \cap \mathbf{F}_3)\} = P\{(\mathbf{F}_1 \cap \mathbf{F}_2' \cap \mathbf{F}_3) + P(\mathbf{F}_1' \cap \mathbf{F}_2 \cap \mathbf{F}_3)\} =$

$(1/6)(5/6)(1/6) + (5/6)(1/6)(1/6) = 10/216$

Step 3: Since $P\{X = 2\} + P\{X = 3\} + P\{X = 4\} = 1$,

we have $P\{X = 4\} = 1 - 1/36 - 10/216 = 200/216$.

The following is the table for computing the mean:

X	**P{X = x}**	**xP{X = x}**
2	1/36	2/36
3	10/216	30/216
4	200/216	800/216
	Total 1	**E(X)** = 832/216

The following is the table for computing the variance σ^2:

X = x	**P{X = x}** = P{**X = x²**}	**x²**	**x²P{X = x²}**
2	1/36	4	24/216
3	10/216	9	90/216
4	200/216	16	3200/216
	Total 1		**E(X²)** = 3314/216

Since $\sigma^2 = E(X^2) - E(X)^2$, from the 2 tables above we have

$\sigma^2 = 3314/216 - (832/216)^2 \approx 0.51$

16.1 - Problem 2:

Let X equal the amount he wins/losses.
$\mathbf{W_k}$: the event he wins the k th game (k = 1,2,3).

$P\{X = \$610\} = P(\mathbf{W_1} \cap \mathbf{W_2} \cap \mathbf{W_3}) = (0.60)^3 = 0.216$

$P\{X = -\$100\} = 1 - 0.216 = 0.784$

X = x	**P{X = x}**	**xP{X = x}**
\$610	0.216	\$131.76
-\$100	0.784	- \$78.40
	Total 1	**E(X)** = \$53.36 average winnings

The table for computing the variance is

X = x	**P{X = x}**	**x²**	**x²P{X = x²}**
\$610	0.216	372100	80373.60
-\$100	0.784	10000	7840.00
			Total = 88213.60
	Total = 1		$\sigma^2 = 88213.60 - 53.36^2 \approx \$85366.31$

Supplementary Problems

1.
A: The event that option A doubles in value.
B: The event that option B doubles in value.
C: The event that option C doubles in value.

Let Z equal her total possible returns.

$P\{Z = -\$1{,}500\} = P(\mathbf{A}' \cap \mathbf{B}' \cap \mathbf{C}') = (0.30)(0.40)(0.45) = 0.054$

$P\{Z = -\$500\} = P[(\mathbf{A} \cap \mathbf{B}' \cap \mathbf{C}') \cup (\mathbf{A}' \cap \mathbf{B} \cap \mathbf{C}') \cup (\mathbf{A}' \cap \mathbf{B}' \cap \mathbf{C})] =$

P(**A**∩**B**′∩**C**′)+ P(**A**′∩**B**∩**C**′)+ P(**A**′∩**B**′∩**C**)] =

(0.70)(0.40)(0.45) + (0.30)(0.60)(0.45) + (0.30)(0.40)(0.55) = 0.126 + 0.081 + 0.066 = 0.273

P{Z = $500} = P[(**A**∩**B**∩**C**′)∪(**A**∩**B**′∩**C**)∪(**A**′∩**B**∩**C**)] =

P(**A**∩**B**∩**C**′)+ P(**A**∩**B**′∩**C**)+ P(**A**′∩**B**∩**C**)] = (0.70)(0.60)(0.45) + (0.70)(0.40)(0.55) +

(0.30)(0.60)(0.55) = 0.189 + 0.154 + 0.099 = 0.442

P{Z = $1,500} = P(**A**∩**B**∩**C**) = (0.70)(0.60)(0.55) = 0.231

Z = z	**P{Z = z}**	**zP{Z = z}**
- $1500	0.054	- $81.00
- $500	0.273	- $136.50
$500	0.442	$221.00
$1500	0.231	$346.50
	Total 1	**E(Z)** = $350 average gain

The table for computing the variance is

X = x	**P{X = x}**	**x^2**	**x^2P{X = x}**
-$1500	0.054	2250000	121500
-$500	0.273	250000	68250
$500	0.442	250000	110500
$1500	0.231	2250000	519750
			Total = 820000
	Total 1		$\sigma^2 \approx \$820,000 - \$350^2 = \$697,500$

2.
From 14.2 - Example 2, Lesson 14, we have the following distribution:

x	P{X = x}	xP{X = x}
2	1/36	2/36
3	2/36	6/36
4	3/36	12/36
5	4/36	20/36
6	5/36	30/36
7	6/36	42/36
8	5/36	40/36
9	4/36	36/36
10	3/36	30/36
11	2/36	22/36
12	1/36	12/36
	Total 1	**E(X)** = 7

X = x	P{X = x}	x^2	x^2P{X = x}
2	1/36	4	4/36
3	2/36	9	18/36
4	3/36	16	48/36
5	4/36	25	100/36
6	5/36	36	180/36
7	6/36	49	294/36
8	5/36	64	320/36
9	4/36	81	324/36
10	3/36	100	300/36
11	2/36	121	242/36
12	1/36	144	144/36
			Total $\frac{1974}{36}$

$$\sigma^2 \approx \frac{1974}{36} - 7^2 = \frac{1974}{36} - \frac{1764}{36} = \frac{210}{36} = \frac{35}{6}$$

3.

➤a.

The following table computes the variance of X:

X = x	**P{X = x}**	**x^2**	**x^2P{X = x}**
\$900	0.60	810000	486000
\$1500	0.30	2250000	675000
\$2000	0.10	4000000	400000
			Total = 1561000
	Total = 1		σ^2_X = \$1561000 - 1190^2 = \$144,900

The following table computes the variance of Y:

Y = y	**P{Y = y}**	**y^2**	**y^2P{Y = y}**
\$500	0.70	250000	175000
\$1100	0.25	1210000	302500
\$1500	0.05	2250000	112500
			Total = 590000
	Total 1		σ^2_X = \$590000 -700^2 = \$100,000

➤b.

$\sigma^2_X + \sigma^2_y$ = \$144,900 + \$100,000 = \$244,900

From Supplementary problem 4, Lesson 15:

Z = X + Y = z	**P{Z = z}**	**zP{Z = z}**
\$1400	0.42	\$588
\$2000	0.36	\$720
\$2400	0.03	\$72
\$2500	0.07	\$175
\$2600	0.075	\$195
\$3000	0.015	\$45
\$3100	0.025	\$77.5
\$3500	0.005	\$17.5
	Total 1	**E(Z)** = \$1,890

Z = z	P{Z = z}	z^2	$z^2P\{Z = z\}$
\$1400	0.42	1960000	823200
\$2000	0.36	4000000	1440000
\$2400	0.03	5760000	172800
\$2500	0.07	6250000	437500
\$2600	0.075	6760000	507000
\$3000	0.015	9000000	135000
\$3100	0.025	9610000	240250
\$3500	0.005	12250000	61250
			Total = 3817000
	Total 1		σ^2_Z = \$3817000 -1890^2 = \$244,900

Therefore, $\sigma^2_Z = \sigma^2_X + \sigma^2_Y$

4.
➤a.
Let X equal the number of possible diamonds drawn.

From Supplementary problem 5, Lesson 15,

X	P{X = x}	xP{X = x}
0	1482/2652	0
1	1014/2652	1014/2652
2	156/2652	312/2652
	Total = 1	$E(X) = \frac{1326}{2652} = 0.5$

X = x	P{X = x}	X^2	x^2P{X = x}
0	1482/2652	0	0
1	1014/2652	1	1014/2652
2	156/2652	4	624/2652
			Total = $\frac{1638}{2652}$

$$\sigma^2_X = \frac{1638}{2652} - 0.5^2 = \frac{325}{884}$$

➤b.

Let Y equal the possible number of clubs drawn.

Since clubs and diamonds are exactly interchangeable,

$$\sigma^2_Y = \frac{325}{884}$$

➤c.

$\mathbf{C_1}$: The first card drawn is a club.
$\mathbf{C_2}$: The second card drawn is a club.

$\mathbf{D_1}$: The first card drawn is a diamond.
$\mathbf{D_2}$: The second card drawn is a diamond.

Let Z = X + Y, the total possible number of diamonds and clubs.

$$P\{Z = 0 + 0 = 0\} = P[(\mathbf{C_1}'\cap\mathbf{D_1}')\cap(\mathbf{C_2}'\cap\mathbf{D_2}')] =$$

$$P(\mathbf{C_1}'\cap\mathbf{D_1}')P[(\mathbf{C_1}'\cap\mathbf{D_1}')(\mathbf{C_2}'\cap\mathbf{D_2}')\,|\,(\mathbf{C_1}'\cap\mathbf{D_1}')] = (\frac{26}{52})(\frac{25}{51}) = \frac{650}{2652}$$

$$P\{Z = 1\} = P[(\mathbf{C_1})\cap(\mathbf{C_2}'\cap\mathbf{D_2}')] + P[(\mathbf{C_2})\cap(\mathbf{C_1}'\cap\mathbf{D_1}')] +$$

$$P[(\mathbf{D_1})\cap(\mathbf{C_2}'\cap\mathbf{D_2}')] + P[(\mathbf{D_2})\cap(\mathbf{C_1}'\cap\mathbf{D_1}')] = (4)(\frac{13}{52})(\frac{26}{51}) = \frac{1352}{2652}$$

$$P\{Z = 2\} = P(\mathbf{C_1}\cap\mathbf{C_2}) + P(\mathbf{D_1}\cap\mathbf{D_2}) + P(\mathbf{D_1}\cap\mathbf{C_2}) + P(\mathbf{C_1}\cap\mathbf{D_2}) =$$

$$(2)(\frac{13}{52})(\frac{12}{51}) + (2)(\frac{13}{52})(\frac{13}{51}) = \frac{650}{2652}$$

Since we are only drawing 2 cards,

P{Z = 3} = 0
P{Z = 4} = 0

Z	**P{Z = z}**	**zP{Z = z}**
0	650/2652	0
1	1352/2652	1352/2652
2	650/2652	1300/2652
3	0	0
4	0	0
	Total = 1	**E(Z) = 1**

X = x	**P{X = x}**	X^2	x^2**P{X = x}**
0	650/2652	0	0
1	1352/2652	1	1352/2652
2	650/2652	4	2600/2652
3	0	9	0
4	0	16	0
	Total 1		$\sigma^2_X = 3952/2652 - 1^2 = 325/663$

$$\sigma^2_X + \sigma^2_Y = \frac{325}{884} + \frac{325}{884} = \frac{650}{884} = \frac{325}{442} \neq \frac{325}{663} = \sigma^2_Z$$

5.
Strategy 1.

$\mathbf{W_1}$: The event that he wins first game.
$\mathbf{W_2}$: The event that he wins second game.
$\mathbf{W_3}$: The event that he wins third game.

Strategy 1.

Let X equal the wins/losses possible.

$P\{X = -\$55 - \$55 - \$55 = -\$165\} = P(\mathbf{W_1}' \cap \mathbf{W_2}' \cap \mathbf{W_3}') = (0.40)^3 = 0.064$

$P\{X = \$50 - \$55 - \$55 = -\$60\} =$

$P(\mathbf{W_1} \cap \mathbf{W_2}' \cap \mathbf{W_3}') + P(\mathbf{W_1}' \cap \mathbf{W_2} \cap \mathbf{W_3}') + P(\mathbf{W_1}' \cap \mathbf{W_2}' \cap \mathbf{W_3}) = 3(0.6)(0.4)^2 = 0.288$

$P\{X = \$50 + \$50 - \$55 = \$45\} =$

$P(\mathbf{W_1} \cap \mathbf{W_2} \cap \mathbf{W_3}') + P(\mathbf{W_1}' \cap \mathbf{W_2} \cap \mathbf{W_3}) + P(\mathbf{W_1} \cap \mathbf{W_2}' \cap \mathbf{W_3}) = 3(0.4)(0.6)^2 = 0.432$

$P\{X = \$50 + \$50 + \$50 = \$150\} = P(\mathbf{W_1} \cap \mathbf{W_2} \cap \mathbf{W_3}) = (0.6)^3 = 0.216$

$E(X) = (-165)(0.064) + (-60)(0.288) + (45)(0.432) + (150)(0.216)$

$= -10.56 \; -17.28 + 19.44 + 32.4 = \$24$

X = x	**P{X = x}**	**X^2**	**x^2P{X = x}**
- $165	0.064	27225	1742.4
- 60	0.288	3600	1036.8
45	0.432	2025	874.8
150	0.216	22500	4860
			Total = 8514
	Total = 1		$\sigma^2_X = 8514 - 24^2 = \$7938$

Strategy 2:

Let X equal the wins/losses possible.

$P\{X = \$130 + \$130 + \$130 = \$390\} = P(\mathbf{W_1} \cap \mathbf{W_2} \cap \mathbf{W_3}) = (0.6)^3 = 0.216$

$P\{X = \$130 - \$100 = \$30\} = P(\mathbf{W_1} \cap \mathbf{W_2} \cap \mathbf{W_3}') + P(\mathbf{W_1} \cap \mathbf{W_2}' \cap \mathbf{W_3}) + P(\mathbf{W_1}' \cap \mathbf{W_2} \cap \mathbf{W_3}) = 3(0.6)^2(0.4)$

$= 0.432$

$P\{X = -\$150\} = 1 - 0.216 - 0.432 = 0.352$

$E(X) = (\$390)(0.216) + (\$30)(0.432) + (-\$150)(0.352) = \$84.24 + 12.96 - 52.8 = \$44.40$

X = x	**P{X = x}**	**X^2**	**x^2P{X = x}**
\$390	0.216	152100	32853.6
30	0.432	900	388.8
-150	0.352	22500	7920.0
			Total 41162.4
	Total 1		σ^2_X = \$41162 - \$$44.4^2$ = \$39191.04

6.

➤a.

Step 1:μ = 16.15

$\sigma^2 = 0.08$

Step 2: If X > 16.25 or X < 16.05 then | X - 16.15| > 0.10 ≥ a = 0.10.

Letting a = 0.10,

we have $P(|X - \mu| \geq a) = P(|X - 16.15| > 0.10) \leq \frac{\sigma^2}{a^2} = \frac{0.08^2}{0.10^2} = 0.64.$

➤b.

Since we are only using X > 16.25, then by symmetry we are only computing half the interval. There the probability is 0.64/2 - 0.32.

7.

➤a.

σ is called the standard deviation.

Let a = 2σ.

$$P(|X - \mu| \geq a) = P(|X - \mu| \geq 2\sigma) \leq \frac{\sigma^2}{(2\sigma)^2} = \frac{1}{4} = 0.25$$

➤b.

Let a = 3σ.

$$P(|X - \mu| \geq a) = P(|X - \mu| \geq 3\sigma) \leq \frac{\sigma^2}{(3\sigma)^2} = \frac{1}{9} \approx 0.11$$

8.
A game is called a "fair game" if E(X) = 0 where X equals the win/losses resulting from the game.

➤a.

X = x	P{X = x}	X^2	x^2P{X = x}
$100	0.523	10000	5230.0
-$110	0.477	12100	5771.7
	Total = 1		σ^2_X = $1101.70 - $0^2 = $1101.70

➤b.

Number of teams in the parley	Winning
2	$ 260
3	$ 600
4	$1,000
5	$2,000

For each of these parleys, p equals the probability of winning each game that the game is fair.

Number of teams in the parley	Winning	P
2	$ 260	0.527
3	$ 600	0.523
4	$1,000	0.549
5	$2,000	0.544

➤Two team parley

X = x	P{X = x}	X^2	x^2P{X = x}
$260	0.278	67600	18792.80
-$100	0.722	10000	7220.00
			Total = 26012.80
	Total 1		σ^2_X = $26012.80 - $0^2 = $26,012.80

➤Three team parley

X = x	**P{X = x}**	**X^2**	**$x^2P\{X = x\}$**
\$600	0.143	360000	51480
-\$100	0.857	10000	8570
			Total 60050
	Total 1		σ^2_X = \$60,050- $\$0^2$ = \$60,050

➤Four team parley

X = x	**P{X = x}**	**X^2**	**$x^2P\{X = x\}$**
\$1000	0.09	1000000	\$90000
-\$100	0.91	10000	\$ 9100
			Total = 99100
	Total 1		σ^2_X = \$99,100- $\$0^2$ = \$99,100

➤Five team parley

X = x	**P{X = x}**	**X^2**	**$x^2P\{X = x\}$**
\$2000	0.048	4000000	\$192,000
-\$100	0.952	10000	\$ 9,520
			Total = \$201,520
	Total 1		σ^2_X = \$201,520- $\$0^2$ = \$201,520

9.

$\sigma_{X+c}{}^2 = E[(X + c)^2] - [E(X + c)]^2 = E(X^2) + E(2cX) + E(c^2) - [E(X) + E(c)]^2 =$

$E(X^2) + 2cE(X) + E(c^2) - \{[E(X)]^2 + 2cE(X) + [E(c)]^2\} =$

$E(X^2) + 2cE(X) + c^2 - [E(X)]^2 - 2cE(X) - c^2 =$

$E(X^2) - [E(X)]^2 = \sigma_X{}^2$

10.

$$E(Z) = \frac{E(X-\mu)}{E(\sigma)} = \frac{0}{E(\Sigma)} = 0$$

$$\sigma_Z^2 = E(Z)^2 - [E(Z)]^2 = E[\frac{(X - \mu)}{\sigma}]^2 - [E(\frac{X - \mu}{\sigma})]^2 =$$

$$E(\frac{X^2 - 2X\mu + \mu^2}{\sigma^2}) - 0 = \frac{[EX^2 - 2\mu^2 + \mu^2]}{\sigma^2} = \frac{\sigma^2}{\sigma^2} = 1$$

11.
$E(X) = 1p + 0(1 - p) = p$

$\sigma^2 = E(X^2) - [E(X)]^2$

$E(X^2) = 1^2p + 0(1 - p) = p$

$\sigma^2 = p - p^2 = p(1 - p)$

12.
➤a.
$\sigma^2 = P\{X = x_1\}[x_1 - E(X)]^2 + P\{X = x_2\}[x_2 - E(X)]^2 + ... + P\{X = x_n\}(x_n - E(X))^2 =$

$P\{X = x_1\}[x_1^2 - 2x_1E(X) + E(X)^2] + P\{X = x_2\}[x_2^2 - 2x_2E(X) + E(X)^2] + ... +$

$P\{X = x_n\}[x_n^2 - 2x_nE(X) + E(X)^2] =$

$x_1^2P\{X = x_1\} + x_2^2P\{X = x_2\}+...+ x_n^2P\{X = x_n\} - [2x_1E(X)P\{X = x_1\}+ ... + 2x_nE(X)P\{X = x_n\}] +$

$E(X)^2P\{X = x_1\} + E(X)^2P\{X = x_2\} + ... + E(X)^2P\{X = x_n\} =$

$E(X^2) - 2E(X)E(X) + E(X)^2 = E(X^2) - E(X)^2$

➤b.
From a, we have $\sigma^2 = E(X^2) - E(X)^2$.

Therefore, $E(X^2) = \sigma^2 + E(X)^2 = \sigma^2 + \mu^2$.

13.
➤a.
Step 1: First we show S_2, X_k are independent ($k > 2$).

$P\{S_2 = x_1 + x_2; X_3 = x_3\} = P[\{X_1 = x_1\}\cap\{X_2 = x_2\}\cap\{X_{n+1} = x_3\}] =$

$P\{X_1 = x_1\}P\{X_2 = x_2\}P\{X_3 = x_3\} = P[\{X_1 = x_1\}\cap\{X_2 = x_2\}]P\{X_3 = x_3\}] =$

$P\{X_1 = x_1 ;X_2 = x_2\}P\{X_3 = x_3\} = P\{S_2 = x_1 + x_2\}P\{X_3 = x_3\}$

Step 2: Using induction, we assume S_{n-1}, X_n ,X_{n+1} are independent.

$P\{S_{n-1} + X_n= s_{n-1} + x_n ; X_{n+1} = x_{n+1}\} = P\{S_n = s_n ; X_{n+1} = x_{n+1}\} =$

$P\{S_{n-1} = s_{n-1}; X_n = x_n; X_{n+1} = x_{n+1}\} = P[\{S_{n-1} = s_{n-1}\}\cap\{X_n = x_n\}\cap\{X_{n+1} = x_{n+1}\}] =$

$P\{S_{n-1} = s_{n-1}\}P\{X_n = x_n\}P\{X_{n+1} = x_{n+1}\} =$

$P[\{S_{n-1} = s_{n-1}\} \cap \{X_n = x_n\}]\, P\{X_{n+1} = x_{n+1}\} = P\{S_n = s_{n-1} + x_n\} P\{X_{n+1} = x_{n+1}\}$

➤b.
$\sigma^2(S_2) = \sigma^2(X_1 + X_2) = E[(X_1 + X_2)^2] - [E(X_1 + X_2)]^2 =$

$E(X_1^2 + 2X_1X_2 + X_2^2) - [E(X_1) + E(X_2)]^2 =$

$E(X_1^2) + 2E(X_1X_2) + E(X_2^2) - [E(X_1)^2 + 2E(X1)E(X_2) + E(X_2)^2] =$

$E(X_1^2) + 2E(X_1)E(X_2) + E(X_2^2) - [E(X_1)^2 + 2E(X1)E(X_2) + E(X_2)^2] =$

$E(X_1^2) + E(X_2^2) - E(X_1)^2 - E(X_2)^2 = [E(X_1^2) - E(X_1)^2] + [E(X_2^2) - E(X_2)^2] =$

$\sigma^2(X_1) + \sigma^2(X_2) = \sigma^2(S_2)$

By induction, we assume $\sigma^2(S_n) = \sigma^2(X_1) + \ldots + \sigma^2(X_n)$

By Problem a. we know S_n and X_{n+1} are independent. Therefore,

$\sigma^2(S_{n+1}) = \sigma^2(S_n + X_{n+1}) + \sigma^2(S_n) + \sigma^2(X_{n+1}) = \sigma^2(X_1) + \ldots + \sigma^2(X_{n+1})$

➤c.
$\sigma^2(cX) = E[(cX)^2] - E[(cX)]^2 = c^2E[X^2] - c^2[E(X)]^2$

➤d.
$$\sigma^2(\overline{X}) = \sigma^2\left(\frac{X_1 + X_2 + \ldots + X_n}{n}\right) = \sigma^2\left(\frac{X_1}{n}\right) + \ldots + \sigma^2\left(\frac{X_n}{n}\right) = \frac{\sigma^2(X_1)}{n^2} + \ldots + \frac{\sigma^2(X_n)}{n^2} =$$

$$\frac{\sigma^2 + \ldots + \sigma^2}{n^2} = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

➤e.
Step 1: $nS^2 = (X_1 - \overline{X})^2 + (X_2 - \overline{X})^2 + \ldots + (X_n - \overline{X})^2$

Step 2: $E(nS^2) = E(X_1 - \overline{X})^2 + E(X_2 - \overline{X})^2 + \ldots + E(X_n - \overline{X})^2$

Step 3: $E(X_1 - \overline{X})^2 = E[X_1^2 - 2X_1\overline{X} + \overline{X}^2] = E[(X_1)^2] - 2E(X_1\overline{X}) + E(\overline{X}^2)$

Step 4: $E(X_1^2) = \sigma^2 + \mu^2$

$$E(X_1\overline{X}) = E\left(\frac{X_1X_1 + X_1X_2 + \ldots + X_1X_n}{n}\right) = \frac{E(X_1^2) + E(X_1X_2) + \ldots + E(X_1X_n)}{n} =$$

$$\frac{E(X_1^2) + E(X_1)E(X_2) + ... + E(X_1)E(X_n)}{n} = \frac{E(X_1)^2 + \mu^2 + ... + \mu^2}{n} = \frac{E(X_1^2) + (n-1)\mu^2}{n}$$

$$2E(X_1\overline{X}) = \frac{2[E(X_1^2) + (n-1)\mu^2)]}{n} = \frac{2[\sigma^2 + \mu^2 + (n-1)\mu^2]}{n} = \frac{2[\sigma^2 + n\mu^2]}{n}$$

$$E(\overline{X}^2) = \sigma^2(\overline{X}) + [E(\overline{X})]^2 = \frac{\sigma^2}{n} + \mu^2$$

Step 5: $E(X_1 - \overline{X})^2 = \sigma^2 + \mu^2 \frac{-2[\sigma^2 + n\mu^2]}{n} + \frac{\sigma^2}{n} + \mu^2$

Since each X_1 - $\bar{x}$ has the same distribution, we sum each term if $E(nS^2)$ and get

$$E(nS^2) = E(X_1 - \overline{X})^2 + E(X_2 - \overline{X})^2 +... + E(X_n - \overline{X})^2 =$$

$nE(S^2) = n[\sigma^2 + \mu^2 \frac{-2[\sigma^2 + n\mu^2]}{n} + \frac{\sigma^2}{n} + \mu^2] = n\sigma^2 + n\mu^2 - 2[\sigma^2 + n\mu^2] + \sigma^2 + n\mu^2 = (n-1)\sigma^2$

Therefore, $E(S^2) = [(n - 1)/n]\ \sigma^2$.

14.
From lesson 15, $P\{X = k\} = [k^3 - (k-1)^3]216^{-1}$, $k = 1,..,6$ and $E(X) \approx 4.96$.

$\sigma^2 = E(X^2) - E(X)^2$

$P(X^2 = k^2) = P\{X = k\} = [k^3 - (k-1)^3]216^{-1}$, $k = 1,..,6$.

$E(X^2) = 1^2[1^3 - (1-1)^3]216^{-1} + 2^2[2^3 - (2-1)^3]216^{-1} + 3^2[3^3 - (3-1)^3]216^{-1} +$

$4^2[4^3 - (4-1)^3]216^{-1} + 5^2[5^3 - (5-1)^3]216^{-1} + 6^2[6^3 - (6-1)^3]216^{-1} =$

$[1 + 4(7) + 9(19) + 16(37) + 25(61) + 36(91)]216^{-1} = [1 + 28 + 171 + 592 + 1525 + 3276]216^{-1} \approx$

25.89

$\sigma^2 = E(X^2) - E(X)^2 = 25.89 - 4.96^2 \approx 1.29$

15.
$\mathbf{H}_k$: head occurs on the kth toss.
$\mathbf{K}_k$: tails occurs on the kth toss.

$\{X = 1\} = \mathbf{H}_1$
$\{X = 2\} = \mathbf{T}_1 \cap \mathbf{H}_2$

$\{X = 3\} = \mathbf{T}_1 \cap \mathbf{T}_2 \cap \mathbf{H}_3$

$\{X = 4\} = \mathbf{T}_1 \cap \mathbf{T}_2 \cap \mathbf{T}_3 \cap \mathbf{H}_4$

$\{X = 5\} = (\mathbf{T}_1 \cap \mathbf{T}_2 \cap \mathbf{T}_3 \cap \mathbf{T}_4 \cap \mathbf{H}_5) \cup (\mathbf{T}_1 \cap \mathbf{T}_2 \cap \mathbf{T}_3 \cap \mathbf{T}_4 \cap \mathbf{T}_5)$

$P\{X = 1\} = P(\mathbf{H}_1) = 1/2$

$P\{X = 2\} = P(\mathbf{T}_1 \cap \mathbf{H}_2) = P(\mathbf{T}_1)P(\mathbf{H}_2) = (1/2)(1/2) = 1/4$

$P\{X = 3\} = \mathbf{P}(\mathbf{T}_1 \cap \mathbf{T}_2 \cap \mathbf{H}_3) = \mathbf{P}(\mathbf{T}_1)P(\mathbf{T}_2)P(\mathbf{H}_3) = (1/2)(1/2)(1/2) = 1/8$

$P\{X = 4\} = P(\mathbf{T}_1 \cap \mathbf{T}_2 \cap \mathbf{T}_3 \cap \mathbf{H}_4) = P(\mathbf{T}_1)P(\mathbf{T}_2)P(\mathbf{T}_3)P(\mathbf{H}_4) = (1/2)(1/2)(1/2)(1/2) = 1/16$

$P\{X = 5\} = P\{(\mathbf{T}_1 \cap \mathbf{T}_2 \cap \mathbf{T}_3 \cap \mathbf{T}_4 \cap \mathbf{H}_5) \cup (\mathbf{T}_1 \cap \mathbf{T}_2 \cap \mathbf{T}_3 \cap \mathbf{T}_4 \cap \mathbf{T}_5)\} =$

$P(\mathbf{T}_1)P(\mathbf{T}_2)P(\mathbf{T}_3)P(\mathbf{T}_4)P(\mathbf{H}_5) + P(\mathbf{T}_1)P(\mathbf{T}_2)P(\mathbf{T}_3)P(\mathbf{T}_4)P(\mathbf{T}_5) = 1/32 + 1/32 = 1/16$

Step 1: $E(X) = 1(1/2) + 2(1/4) + 3(1/8) + 4(1/16) + 5(1/16) = 1/2 + 2/4 + 3/8 + 4/16 + 5/16 = 31/16$

Step 2: $P\{X^2 = 1^2 = 1\} = P(\mathbf{H}_1) = 1/2$

$P\{X^2 = 2^2 = 4\} = P(\mathbf{T}_1 \cap \mathbf{H}_2) = (1/2)(1/2) = 1/4$

$P\{X^2 = 3^2 = 9\} = \mathbf{P}(\mathbf{T}_1 \cap \mathbf{T}_2 \cap \mathbf{H}_3) = (1/2)(1/2)(1/2) = 1/8$

$P\{X^2 = 4^2 = 16\} = P(\mathbf{T}_1)P(\mathbf{T}_2)P(\mathbf{T}_3)P(\mathbf{H}_4) = (1/2)(1/2)(1/2)(1/2) = 1/16$

$P\{X^2 = 5^2 = 25\} = P(\mathbf{T}_1)P(\mathbf{T}_2)P(\mathbf{T}_3)P(\mathbf{T}_4)P(\mathbf{H}_5) + P(\mathbf{T}_1)P(\mathbf{T}_2)P(\mathbf{T}_3)P(\mathbf{T}_4)P(\mathbf{T}_5) = 1/16$

$E(X^2) = 1(1/2) + 4(1/4) + 9(1/8) + 16(1/16) + 25(1/16) = 1/2 + 4/4 + 9/8 + 16/16 + 25/16 = 83/16$

$\sigma^2 = E(X^2) - [E(X)]^2 = 83/16 - (31/16)^2 = 1328/256 - 961/256 = 367/256$

16.

$\mathbf{R}_k$: The kth marble selected is red.

$\mathbf{B}_k$: The kth marble selected is black.

$\mathbf{S} = (X = 2) \cup (X = 3) \cup (X = 4)$

Step 1:

$(X = 2) = (\mathbf{R}_1 \cap \mathbf{B}_2) \cup (\mathbf{B}_1 \cap \mathbf{R}_2)$

$(X = 3) = (\mathbf{R}_1 \cap \mathbf{R}_2 \cap \mathbf{B}_3) \cup (\mathbf{B}_1 \cap \mathbf{B}_2 \cap \mathbf{R}_3)$

$(X = 4) = (\mathbf{R}_1 \cap \mathbf{R}_2 \cap \mathbf{R}_3 \cap \mathbf{B}_4) \cup (\mathbf{B}_1 \cap \mathbf{B}_2 \cap \mathbf{B}_3 \cap \mathbf{R}_4)$

Step 2:
P(X = 2) =

$P(\mathbf{R}_1 \cap \mathbf{B}_2) + P(\mathbf{B}_1 \cap \mathbf{R}_2) = P(\mathbf{R}_1)P(\mathbf{B}_2|\mathbf{R}_1) + P(\mathbf{B}_1)P(\mathbf{R}_2|\mathbf{B}_1) = (3/6)(3/5) + (3/6)(3/5) = 18/30$

P(X = 3) =

$P(\mathbf{R}_1 \cap \mathbf{R}_2 \cap \mathbf{B}_3) + P(\mathbf{B}_1 \cap \mathbf{B}_2 \cap \mathbf{R}_3) = P(\mathbf{R}_1)P(\mathbf{R}_2|\mathbf{R}_1)P(\mathbf{B}_3|\mathbf{R}_1 \cap \mathbf{R}_2) + P(\mathbf{B}_1)P(\mathbf{B}_2|\mathbf{B}_1)P(\mathbf{R}_3|\mathbf{B}_1 \cap \mathbf{B}_2) =$

(3/6)(2/5)(3/4) + (3/6)(2/5)(3/4) = 36/120

P(X = 4) =

$P(\mathbf{R}_1 \cap \mathbf{R}_2 \cap \mathbf{R}_3 \cap \mathbf{B}_4) + P(\mathbf{B}_1 \cap \mathbf{B}_2 \cap \mathbf{B}_3 \cap \mathbf{R}_4)$

Since P(**S**) = P(X = 2) + P(X = 3) + P(X = 4) = 1,

18/30 + 36/120 + P(X = 4) = 1

P(X = 4) = 1 - 18/30 - 36/120 = 1 - 108/120 = 12/120 = 1/10

Step 3: E(X) = 2(18/30) + 3(36/120) + 4(1/10) = 2(6/10) + 3(3/10) + 4(1/10) = 25/10 = 5/2 = 2.5

Step 4: $(X^2 = 4) = (\mathbf{R}_1 \cap \mathbf{B}_2) \cup (\mathbf{B}_1 \cap \mathbf{R}_2)$

$(X^2 = 9) = (\mathbf{R}_1 \cap \mathbf{R}_2 \cap \mathbf{B}_3) \cup (\mathbf{B}_1 \cap \mathbf{B}_2 \cap \mathbf{R}_3)$

$(X^2 = 16) = (\mathbf{R}_1 \cap \mathbf{R}_2 \cap \mathbf{R}_3 \cap \mathbf{B}_4) \cup (\mathbf{B}_1 \cap \mathbf{B}_2 \cap \mathbf{B}_3 \cap \mathbf{R}_4)$

$P(X^2 = 4) = 18/30$

$P(X^2 = 9) = 36/120$

$P(X^2 = 16) = 1/10$

$E(X^2) = 4(18/30) + 9(36/120) + 16(1/10) = 72/30 + 324/120 + 16/10 = 67/10$

$\sigma^2 = E(X^2) - E(X)^2 = 67/10 - (5/2)^2 = 67/10 - 25/4 = 9/20$

17.
$\mathbf{A}_k$: urn A is selected. on the k th toss (k = 1,2).
$\mathbf{B}_k$: urn B is selected on the k th toss. (k = 1,2).
$\mathbf{R}_k$: red marble is selected on the k th toss (k = 1,2).

$\mathbf{W}_k$: white marble is selected on the k th toss. (k = 1,2).

Step 1: $\mathbf{S} = (X = 1)\cup(X = 2)\cup(X = 3)$, since the maximum number of selections is 3.

$(X = 1) = (\mathbf{A}_1\cap\mathbf{R}_1)\cup(\mathbf{B}_1\cap\mathbf{R}_1)$

$(X = 2) = [(\mathbf{A}_1\cap\mathbf{W}_1)\cup(\mathbf{B}_1\cap\mathbf{W}_1)]\cap[(\mathbf{A}_2\cap\mathbf{R}_2)\cup(\mathbf{B}_2\cap\mathbf{R}_2)] =$

$\{(\mathbf{A}_1\cap\mathbf{W}_1)\cap[(\mathbf{A}_2\cap\mathbf{R}_2)\cup(\mathbf{B}_2\cap\mathbf{R}_2)]\}\cup\{(\mathbf{B}_1\cap\mathbf{W}_1)\cap[(\mathbf{A}_2\cap\mathbf{R}_2)\cup(\mathbf{B}_2\cap\mathbf{R}_2)]\}=$

$[(\mathbf{A}_1\cap\mathbf{W}_1)\cap(\mathbf{A}_2\cap\mathbf{R}_2)]\cup[(\mathbf{A}_1\cap\mathbf{W}_1)\cap(\mathbf{B}_2\cap\mathbf{R}_2)]\cup[(\mathbf{B}_1\cap\mathbf{W}_1)\cap(\mathbf{A}_2\cap\mathbf{R}_2)]\cup[(\mathbf{B}_1\cap\mathbf{W}_1)\cap(\mathbf{B}_2\cap\mathbf{R}_2)]$

Step 2: $P(X = 1) = P(\mathbf{A}_1\cap\mathbf{R}_1) + P(\mathbf{B}_1\cap\mathbf{R}_1) = P(\mathbf{A}_1)P(\mathbf{R}_1|\mathbf{A}_1) + P(\mathbf{B}_1)P(\mathbf{R}_1|\mathbf{B}_1) =$

$(1/2)(1/2) + (1/2)(1/2)= 1/2.$

$P(X = 2) =$

$P[(\mathbf{A}_1\cap\mathbf{W}_1)\cap(\mathbf{A}_2\cap\mathbf{R}_2)] + P[(\mathbf{A}_1\cap\mathbf{W}_1)\cap(\mathbf{B}_2\cap\mathbf{R}_2)] + P[(\mathbf{B}_1\cap\mathbf{W}_1)\cap(\mathbf{A}_2\cap\mathbf{R}_2)] + P[(\mathbf{B}_1\cap\mathbf{W}_1)\cap(\mathbf{B}_2\cap\mathbf{R}_2)] =$

$P(\mathbf{A}_1\cap\mathbf{W}_1)P(\mathbf{A}_2\cap\mathbf{R}_2|\mathbf{A}_1\cap\mathbf{W}_1) + P(\mathbf{A}_1\cap\mathbf{W}_1)P(\mathbf{B}_2\cap\mathbf{R}_2|\mathbf{A}_1\cap\mathbf{W}_1) +$

$P(\mathbf{B}_1\cap\mathbf{W}_1)P(\mathbf{A}_2\cap\mathbf{R}_2|\mathbf{B}_1\cap\mathbf{W}_1) + P(\mathbf{B}_1\cap\mathbf{W}_1)P(\mathbf{B}_2\cap\mathbf{R}_2|\mathbf{B}_1\cap\mathbf{W}_1)$

From lesson 13, problem 9, we have

$P(\mathbf{A}_2\cap\mathbf{R}_2|\mathbf{A}_1\cap\mathbf{W}_1) = P(\mathbf{A}_2|\mathbf{A}_1\cap\mathbf{W}_1)\ P[\mathbf{R}_2|(\mathbf{A}_1\cap\mathbf{W}_1)\cap\mathbf{A}_2] = (1/2)(1) = 1/2$

$P(\mathbf{B}_2\cap\mathbf{R}_2|\mathbf{A}_1\cap\mathbf{W}_1) = P(\mathbf{A}_2|\mathbf{A}_1\cap\mathbf{W}_1)\ P[\mathbf{R}_2|(\mathbf{A}_1\cap\mathbf{W}_1)\cap\mathbf{B}_2] = (1/2)(1/2) = 1/4$

$P(\mathbf{A}_2\cap\mathbf{R}_2|\mathbf{B}_1\cap\mathbf{W}_1) = P(\mathbf{A}_2|\mathbf{B}_1\cap\mathbf{W}_1)\ P[\mathbf{R}_2|(\mathbf{B}_1\cap\mathbf{W}_1)\cap\mathbf{A}_2] = (1/2)(1/2) = 1/4$

$P(\mathbf{B}_2\cap\mathbf{R}_2|\mathbf{B}_1\cap\mathbf{W}_1) = P(\mathbf{B}_2|\mathbf{B}_1\cap\mathbf{W}_1)\ P[\mathbf{R}_2|(\mathbf{B}_1\cap\mathbf{W}_1)\cap\mathbf{B}_2] = (1/2)(1) = 1/2$

$P(X = 2) =$

$P(\mathbf{A}_1\cap\mathbf{W}_1)P(\mathbf{A}_2\cap\mathbf{R}_2|\mathbf{A}_1\cap\mathbf{W}_1) + P(\mathbf{A}_1\cap\mathbf{W}_1)P(\mathbf{B}_2\cap\mathbf{R}_2|\mathbf{A}_1\cap\mathbf{W}_1) +$

$P(\mathbf{B}_1\cap\mathbf{W}_1)P(\mathbf{A}_2\cap\mathbf{R}_2|\mathbf{B}_1\cap\mathbf{W}_1) + P(\mathbf{B}_1\cap\mathbf{W}_1)P(\mathbf{B}_2\cap\mathbf{R}_2|\mathbf{B}_1\cap\mathbf{W}_1) =$

$P(\mathbf{A}_1)P(\mathbf{W}_1|\mathbf{A}_1)P(\mathbf{A}_2\cap\mathbf{R}_2|\mathbf{A}_1\cap\mathbf{W}_1) + P(\mathbf{A}_1)P(\mathbf{W}_1|\mathbf{A}_1)P(\mathbf{B}_2\cap\mathbf{R}_2|\mathbf{A}_1\cap\mathbf{W}_1) +$

$P(\mathbf{B}_1)P(\mathbf{W}_1|\mathbf{B}_1)P(\mathbf{A}_2\cap\mathbf{R}_2|\mathbf{B}_1\cap\mathbf{W}_1) + P(\mathbf{B}_1)P(\mathbf{W}_1|\mathbf{B}_1)P(\mathbf{B}_2\cap\mathbf{R}_2|\mathbf{B}_1\cap\mathbf{W}_1) =$

(1/2)(1/2)(1/2) + (1/2)(1/2)(1/4) + (1/2)(1/2)(1/4) + (1/2)(1/2)(1/2) =

1/8 + 1/16 + 1/16 + 1/8 = 6/16 = 3/8

Step 3: **S** = $(X = 1)\cup(X = 2)\cup(X = 3)$, since only a maximum of 3 selections are possible.

P(**S**) = P(X = 1) + P(X = 2) + P(X = 3) = 1

P(X = 3) = 1 - P(X = 1) - P(X = 2) = 1 - 1/2 - 3/8 = 1/8

Step 4: E(X) = 1(1/2) + 2(3/8) + 3(1/8) = 13/8

Step 5: $E(X^2) = 1^2(1/2) + 2^2(3/8) + 3^2(1/8) = (1/2) + 4(3/8) + 9(1/8) = 25/8$

$\sigma^2 = E(X^2) - E(X)^2 = 25/8 - (13/8)^2 = 31/64$

18.
$\text{Cov}(X,Y) = E[(X - \mu_X)(Y - \mu_Y)] = E[XY - \mu_Y X - \mu_X Y + \mu_X\mu_Y] =$

$E(XY) - E(\mu_Y X) - E(\mu_X Y) + E(\mu_X\mu_Y) = E(XY) - \mu_Y E(X) - \mu_X E(Y) + E(\mu_X\mu_Y) =$

$E(XY) - \mu_Y \mu_X - \mu_X \mu_Y + \mu_X\mu_Y = E(XY) - \mu_X \mu_Y$

19.
Since X,Y are independent, we have shown (Lesson 15, supplementary problems)

$E(XY) = E(X)E(Y) = \mu_X \mu_Y$

By problem 18, we have

$\text{Cov}(X,Y) = E(XY) - \mu_X \mu_Y = E(X)E(Y) - \mu_X \mu_Y = \mu_X \mu_Y - \mu_X \mu_Y = 0.$

20.
Define $Z = X_1 + X_2$ and $W = X_1 - X_2$.

➤a.
Step 1: **S** = {(h,h), (t,t) (h,t), (t,h)}

$E(X_1) = 1(1/2) + 0(1/2) = 1/2$
$E(X_2) = 1(1/2) + 0(1/2) = 1/2$
$E(X_1^2) = 1^2(1/2) + 0^2(1/2) = 1/2$
$E(X_2^2) = 1^2(1/2) + 0^2(1/2) = 1/2$

$\mu_Z = E(Z) = E(X_1 + X_2) = E(X_1) + E(X_2) = 1/2 + 1/2 = 1$
$\mu_W = E(W) = E(X_1 - X_2) = E(X_1) - E(X_2) = 1/2 - 1/2 = 0$

Step 2: $ZW = (X_1 + X_2)(X_1 - X_2) = X_1^2 - X_2^2$

$E(ZW) = E(X_1^2 - X_2^2) = E(X_1^2) - E(X_2^2) = 1/2 - 1/2 = 0$

$Cov(Z,W) = E(ZW) - \mu_Z\mu_W = 0 - 1(0) = 0.$

➤b.
$\{Z = X_1 + X_2 = 2\} = \{(h,h)\}$

$\{W = X_1 - X_2 = 1\} = \{(h,t)\}$

Since $\{W = X_1 - X_2 = 1\}\cap\{Z = X_1 + X_2 = 2\} = \phi$, $P\{\{Z = X_1 + X_2 = 2\}|\{W = X_1 - X_2 = 1\}\} = \phi$
Z and W are dependent.

21.
The results of problems 19, and 20 show the following:
If 2 random variables are independent, then the covariance will equal 0. However, the converse is not true. In problem 20, random variables Z,W have covariance 0 but they are not independent.

22.
Step 1:
X = 0,1,2; possible number of diamonds drawn.
Y = 0,1,2; possible number of clubs drawn.
XY = 0,1; possible since there are only 2 cards selected.
$\mathbf{D}_1$: diamond drawn on the first drawing.
$\mathbf{D}_2$: diamond drawn on the second drawing.
$\mathbf{C}_1$: club drawn on the first drawing.
$\mathbf{C}_2$: club drawn on the second drawing.

$(XY = 1) = (\mathbf{D}_1\cap\mathbf{C}_2)\cup(\mathbf{C}_1\cap\mathbf{D}_2)$

$P(XY = 1) = P[(\mathbf{D}_1\cap\mathbf{C}_2)\cup(\mathbf{C}_1\cap\mathbf{D}_2)] = P(\mathbf{D}_1\cap\mathbf{C}_2) + P(\mathbf{C}_1\cap\mathbf{D}_2) = (13/52)(13/51) + (13/52)(13/51) =$

$338/2652$

$P(XY = 0) = 1 - P(XY = 1) = 1 - 338/2652 = 2314/2652$

Step 2: $E(XY) = 0(2314/2652) + 1(338/2652) = 338/2652$

$(X = 0) = (\mathbf{D}_1'\cap\mathbf{D}_2')$
$(X = 1) = (\mathbf{D}_1\cap\mathbf{D}_2')\cup(\mathbf{D}_1'\cap\mathbf{D}_2)$

$(X = 2) = (\mathbf{D}_1 \cap \mathbf{D}_2)$

$P(X = 0) = P(\mathbf{D}_1' \cap \mathbf{D}_2') = (39/52)(38/51) = 1482/2652$

$P(X = 1) = P[(\mathbf{D}_1 \cap \mathbf{D}_2') \cup (\mathbf{D}_1' \cap \mathbf{D}_2)] = P[(\mathbf{D}_1 \cap \mathbf{D}_2') + P(\mathbf{D}_1' \cap \mathbf{D}_2)] = (13/52)(39/51) + (39/52)(13/51) = (1014)/(2652)$

$P(X = 2) = P(\mathbf{D}_1 \cap \mathbf{D}_2) = (13/52)(12/51) = 156/2652$

$\mu_X = E(X) = 0P(X = 0) + 1P(X = 1) + 2P(X = 2) = 1014/2652 + 312/2652 = 1326/2652 = 1/2$

Using the same argument,

$\mu y = E(Y) = 0P(Y = 0) + 1P(Y = 1) + 2P(Y = 2) = 1014/2652 + 312/2652 = 1326/2652 = 1/2$

$Cov(X,Y) = E(XY) - \mu_X \mu_y = 338/2652 - (1/2)^2 = 169/1326 - 1/4 = (676 - 1326)/5304 = - 325/2652$

23.

$\sigma^2_{X+Y+Z} = E[(X + Y + Z)^2] - [E(X + Y + Z)]^2$

$E[(X + Y + Z)^2] = E[X^2 + XY + XZ + YX + Y^2 + YZ + ZX + ZY + Z^2] =$

$E(X^2) + E(Y^2) + E(Z^2) + 2E(XY) + 2E(XZ) + 2E(YZ)$

$[E(X + Y + Z)]^2 = [E(X) + E(Y) + E(Z)]^2 =$

$E(X)^2 + E(Y)^2 + E(Z)^2 + 2E(X)E(Y) + 2E(X)E(Z) + 2E(Y)E(Z)$

$\sigma^2_{X+Y+Z} = E[(X + Y + Z)^2] - [E(X + Y + Z)]^2 =$

$E(X^2) + E(Y^2) + E(Z^2) + 2E(XY) + 2E(XZ)+2E(YZ) +$

$- E(X)^2 - E(Y)^2 - E(Z)^2 - 2E(X)E(Y) - 2E(X)E(Z) - 2E(Y)E(Z) =$

$[E(X^2) - E(X)^2] + [E(Y^2) - E(Y)^2] + [E(Z^2) - E(Z)^2] +$

$2\{[E(XY)- 2E(X)E(Y)] + [E(XZ)- E(X)E(Z)] + [E(YZ) - E(Y)E(Z)]\} =$

$\sigma_X^2 + \sigma_Y^2 + \sigma_Z^2 + 2[Cov(X,Y) + Cov(X,Z) + Cov(Y,Z)]$

24.

From problem 23, we have

$\sigma^2_{X+Y+Z} = \sigma_X^2 + \sigma_Y^2 + \sigma_Z^2 + 2[Cov(X,Y) + Cov(X,Z) + Cov(Y,Z)]$

Let Z = 0, we have

$\sigma^2_{X+Y} = \sigma^2_{X+Y+0} = \sigma_X^2 + \sigma_Y^2 + \sigma_0^2 + 2[Cov(X,Y) + Cov(X,0) + Cov(Y,0)] =$

$\sigma_X^2 + \sigma_Y^2 + 2Cov(X,Y)$

Therefore, $Cov(X,Y) = [\sigma^2_{X+Y} - (\sigma_X^2 + \sigma_Y^2)]/2$

25.
Since X,Y are independent, we have

E(XY) = E(X)E(Y) (See Lesson 25, problem 9).

Therefore,

Cov(X,Y) = E(XY) - E(X)E(Y) = E(X)E(Y) - E(X)E(Y) = 0.

Using the same argument,

C(X,Z) = 0 and C(Y,Z) = 0

since each random variables are pair-wise independent.

From , prolem 23, $\sigma^2_{X+Y+Z} = \sigma_X^2 + \sigma_Y^2 + \sigma_Z^2 + 2[Cov(X+Y) + Cov(X+Z) + Cov(Y+Z)] =$

$\sigma^2_{X+Y+Z} = \sigma_X^2 + \sigma_Y^2 + \sigma_Z^2 + 2[0 + 0 + 0] = \sigma_X^2 + \sigma_Y^2 + \sigma_Z^2$

26.
Assume a, b are constants. Show

From problem 24, we have

$Cov(X,Y) = [\sigma^2_{X+Y} - (\sigma_X^2 + \sigma_Y^2)]/2$

$Cov(X+a, Y+b) = [\sigma^2_{(X+a)+(Y+b)} - (\sigma_{X+a}^2 + \sigma_{Y+b}^2)]/2$

From problem 9,

$\sigma^2_{(X+a)+(Y+b)} = \sigma^2_{X+Y}$

$\sigma^2_{X+a} = \sigma^2_X = \sigma_X^2$

$\sigma^2_{Y+b} = \sigma_Y^2$.

Therefore,

$Cov(X + a, Y + b) = . \ [\sigma^2_{X+Y} - \sigma_X^{\ 2} + \sigma_Y^{\ 2})]/2 = Cov(X,Y)$.

27.
Step 1: We first show this is true for special random variables X*, Y* where

$P(X^* = x) = p_1$
$P(X^* = 0) = 1 - p_1$
$P(Y^* = y) = p_2$
$P(Y^* = 0) = 1 - p_2$.

Step 2: Next we show independence of X* and Y*.

Since Cov(X*,Y*) = E(X*Y*) - E(X*)E(Y*) = 0, we have E(X*Y*) = E(X*)E(Y*).

$E(X^*Y^*) = xyP[(X^* = x)\cap(Y^* = y)] + 0x\ P[(X^* = x)\cap(Y^* = 0)] + 0y\ P[(X^* = 0)\cap(Y^* = y)] =$

$xyP[(X^* = x)\cap(Y^* = y)]$

$E(X^*) = xP(X^* = x) + 0P(X^* = 0) = xP(X^* = x)$

$E(Y^*) = yP(Y^* = y) + 0P(Y^* = 0) = yP(X^* = x)$

Since E(X*Y*) = E(X*)E(Y*), we have

$xyP[(X^* = x)\cap(Y^* = y)] = xP(X^* = x)yP(Y^* = y) = xyP(X^* = x)P(Y^* = y)$.

Canceling xy from the above equation we have

$P[(X^* = x)\cap(Y^* = y)] = P(X^* = x)P(Y^* = y)$.

Therefore, the events (X* = x) and (Y* = y) are independent events.

$(X^* = x)' = (X^* = 0)$ and $(Y^* = y)' = (Y^* = 0)$

are the other remaining sets. From lesson 12,

problem 16, these 2 events are also independent.

Step 3: For the general case of the random variable X,Y: $(X = x_1)$, $(X = x_2)$, $(Y = y_1)$, $(Y = y_2)$,

Define $X^* = X - x_1$ and $Y^* = Y - y_1$.

We assume Cov(X,Y) = 0 . Therefore, from problem 26, Cov(X*,Y*) = Cov(X,Y) = 0

Therefore,

$X - x_1$, and $Y - y_1$ are independent.

Step 4: Now $(X = x_1) = (X^* = X - x_1 = 0)$ and $(Y = y_1) = (Y^* = Y - y_1 = 0)$.

Therefore,

$P[(X = x_1)\cap(Y = y_1)] = P[(X^* = 0)\cap(Y^* = 0)] = P(X^* = 0)P(Y^* = 0) = P(X = x_1)P(Y = y_1)$

and independents of X,Y is shown.

28.
From problem 23,

$\sigma^2 x_{1+}x_{2+}x_3 = 3\sigma^2 + 2[3Cov(X_1,X_2)] = 3\sigma^2 + 6Cov(X_1,X_2)$.

Using mathematical induction, we assume the above is true for n -1 and prove it for n :

$\sigma^2 x_{1+}x_{2+}\ldots_+ x_n = n\sigma^2 + n(n-1)Cov(X_1,X_2)$.

Define $S = X_1 + X_2 + \ldots + X_n$

$\sigma^2_S = n\sigma^2 + n(n-1)Cov(X_1,X_2)$.

$\sigma^2 x_{1+}x_{2+}\ldots_+ x_{n+1} = \sigma^2_{(S+\ x_{n+1})} = \sigma^2_S + \sigma^2 x_{n+1} + 2Cov(S,X_{n+1}) =$

$n\sigma^2 + n(n-1)Cov(X_1,X_2) + \sigma^2 x_{n+1} + 2Cov(S,X_{n+1})$

29.
In our solution, we will use the following formulas on summation:

$1 + 2 + \ldots + N = (N)(N + 1)/2$

$1^2 + 2^2 + \ldots + N^2 = N(N + 1)(2N + 1)/6$

X_k : The number selected on the kth ball drawn. (k = 1,2,...,r).

Step 1: Computing the mean of S.

$S = X_1 + X_2 + \ldots + X_r$, the sum of the numbers.

$P(X_k = 1) = 1/10$, where k = 1,2,..., r.

$E(X_k) = 1P(X_k = 1) + 2P(X_k = 2) + ... + 10P(X_k = 10) = 1/10 + 2/10 ... + 10/19 =$

$(1 + 2 + ... + 10)/10 = 10(10 + 1)/2(10) = (11)/2$

$E(S) = E(X_1 + X_2 + ... + X_r) = E(X_1)+ E(X_2) + ... + E(_(X_r) = r(11)/2$

Step 2: Computing the variance of S.

$\sigma^2{}_S = E(S^2) - [E(S)]^2 = E(S^2) - [r(10 + 1)/2]^2 = E(S^2) - r^2(121)/4$

$E(S^2) = E[(X_1 + X_2 + ... + X_r)^2] = E(X_1^2) + E(X_2^2) + ... + E(X_r^2) +$

$E(X_1 X_2) + E(X_1 X_3) + ... + E(X_1 X_r) +$

$E(X_2 X_1) + E(X_2 X_3) + ... + E(X_2 X_r) +$

::

$E(X_r X_1) + E(X_r X_3) + ... + E(X_r X_{r-1})$

$E(X_k^2) = 1^2P(X_k^2 = 1^2) + 2^2 P(X_k^2 = 2^2) + ... + 10^2P(X_k^2 = 10^2) =$

$1^2P(1/10) + 2^2 (1/10) + ... + 10^2 (1/10) = (1^2 + 2^2 + ... + 10^2)(1/10) = [10(10 + 1)(2(10) + 1)/6]/10$

$= (10 + 1)(2(10) + 1)/6$

$E(X_j X_k) = (1)(2)P[X_j = 1;X_k = 2] + (1)(3)P[X_j = 1;X_k = 3] + ... + (1)(10)P[X_j = 1;X_k = 10] +$

$(2)(1)P[X_j = 2;X_k = 1] + (2)(3)P[X_j = 2;X_k = 3] + ... + (1)(10)P[X_j = 2;X_k = 10] + ... +$

$(10)(1)P[X_j = 10;X_k = 1] + (10)(2)P[X_j =10;X_k = 2] + ... + (10)(10 - 1)P[X_j = 10;X_k = 10 - 1] =$

$1\{2/[10(10 - 1)] + 3/ [10(10 - 1)] + ... + 10 /[10(10 - 1)]\} +$

$2\{1/[10(10 - 1)] + 3/[10(10 - 1)] + ... + 10/[10(10 - 1)]\} + ... +$

$10\{1/[10(10 - 1)] + 2/[10(10 - 1)] + ... + (10 - 1)/[10(10 - 1)]\} =$

$1\{2 + 3 + ... + 10\}/[10(10 - 1)]\} + 2\{1 + 3 + .. + 10\}/[10(10 - 1)] + ... +$

$10\{1+ 2 + ... + (10 - 1)\}/[10 (10 - 1)] =$

$1\{ (1 + 2 + 3 + ... + 10) - 1\}/[10(10 - 1)] + 2\{(1 +2 + 3 + .. + 10) - 2\}/[10(10 - 1)] + ... +$

$10\{(1+ 2 + ... + (10 - 1) + 10) - 10 \}/[10 (10 - 1)] =$

$1\{10(10+1)/2 - 1\}/[10(10-1)] + 2\{10(10+1)/2 - 2\}/[10(10-1)] + ... +$

$10\{10(10+1)/2 - 10\}/[10(10-1)] =$

$\{10(10+1)/2 - 1^2\}/[10(10-1)] + \{210(10+1)/2 - 2^2\}/[10(10-1)] + ... +$

$\{(10)(10)(10+1)/2 - 10^2\}/[10(10-1)] =$

$\{(1+2+...+10)10(10+1)/2 - [10(10+1)(2(10)+1)/6]\}/[10(10-1)] =$

$\{10(10+1)/210(10+1)/2 - [10(10+1)(2(10)+1)/6]\}/[10(10-1)] =$

$\{10^2(10+1)^2/4 - [10(10+1)(2(10)+1)/6]\}/[10(10-1)] = E(X_jX_k)$

$E(S^2) = E(X_1^2) + E(X_2^2) + ... + E(X_r^2) +$

$E(X_1X_2) + E(X_1X_3) + ... + E(X_1X_r) +$

$E(X_2X_1) + E(X_2X_3) + ... + E(X_2X_r) +$

::

$E(X_rX_1) + E(X_rX_3) + ... + E(X_rX_{r-1}) =$

$E(S^2) = r(10+1)(2(10)+1)/6 + r^2\{10^2(10+1)^2/4 - [10(10+1)(2(10)+1)/6]\}/[10(10-1)]$

$\sigma^2_S = E(S^2) - [E(S)]^2 = E(S^2) - r^2(10+1)^2/4 =$

$r(10+1)(2(10)+1)/6 + r^2\{10^2(10+1)^2/4 - [10(10+1)(2(10)+1)/6]\}/[10(10-1)] - r^2(10+1)^2/4$

$r(11)(20)+1)/6 + r^2\{100(121)/4 - [10(11)(21)/6]\}/[10(9)] - r^2(11)^2/4 =$

$r231/6 + r^2\{12100/4 - [2310/6]\}/90 - r^2 121/4 =$

$r77/2 + r^2\{3025 - 385\}/90 - r^2 121/4 = r77/2 + r^2\,88/3 - r^2 121/4 = (-11/12)r^2 + (77/2)r$

30.

➤a .

$E(X_k) = 1P(X_k = 1) + 2P(X_k = 2) + 3P(X_k = 3) + 4P(X_k = 4) + 5P(X_k = 5) + 6P(X_k = 6) =$

$1/6 + 2/6 + 3/6 + 4/6 + 5/6 + 6/6 = 3.5$

$$E(\overline{X}) = \frac{E(X_1) + E(X_2) + ... + E(X_{100})}{100} = 100(3.5)/100 = 35/10 = 7/2$$

➤b .

$E(X_k^2) = 1^2P(X = 1^2) + 2^2P(X = 2^2) + 3^2P(X = 3^2) + 4^2P(X = 4^2) + 5^2P(X = 5^2) + 6^2P(X = 6^2) =$

1P (X = 1) + 4P (X = 4) + 9P (X = 9) + 16P (X = 16) + 25P (X = 25) + 36P (X = 36) = 91/6
(k = 1,2,...,100)

$[E(X_k)]^2 = 3.5^2 = 12.25$

$\sigma^2(X_k) = E(X_k^2) - [E(X_k)]^2 = 91/6 - (7/2)^2 = 91/6 - 49/4 = 35/12$

From problem 13d, we have $\sigma^2(\overline{X}) = \frac{\sigma^2}{n}$ = (35/12)(1/100) = 7/240

➤c.

From problem 13, $E(S^2) = \frac{n-1}{n}\sigma^2$.

Since the distribution of the random variables are all equal σ^2, from b. we have $\sigma^2 = 7/240$.

n = 100

Therefore, $E(S^2) = \frac{n-1}{n}\sigma^2$ = (99/100)(7/240) = 33/100(7/80) = 231/8000 ≈ 7/240

31.

➤a.

$X_k = x$	$P(X_k = x)$
0	1/2
1	1/2

➤b.

$\overline{X} = (X_1 + X_2)/2$

$\overline{X} = x$	$P(\overline{X} = x)$
0	1/4
1	1/2
2	1/4

➤c.

$$S^2 = \frac{(X_1 - \overline{X})^2 + (X_2 - \overline{X})^2}{2}$$

$X_1(h,h) = 1$

$X_2(h,h) = 1$

$\overline{X}(h,h) = (1 + 1)/2 = 1$

$S^2 = 0$

$X_1(h,t) = 1$

$X_2(h,t) = 0$

$\overline{X}(h,t) = (1 + 0)/2 = 1/2$

$S^2 = [(1 - 1/2)^2 + (0 - 1/2)^2]/2 = 1/4$

$X_1(t,h) = 0$

$X_2(t,h) = 1$

$\overline{X}(t,h) = (0 + 1)/2 = 1/2$

$S^2 = [(0 - 1/2)^2 + (1 - 1/2)^2]/2 = 1/4$

$X_1(t,t) = 0$

$X_2(t,t) = 0$

$\overline{X}(t,t) = (0 + 0)/2 = 0$

$S^2 = [(0 - 0)^2 + (0 - 0)^2]/2 = 0$

$S^2 = s$	**$P(S^2 = s)$**
0	1/2
1/4	1/2

➤d.

$S^2 = s$	$P(S^2 = s)$	$sP(S^2 = s)$
0	1/2	0
1/4	1/2	1/8
	1	**$E(S^2) = 1/8$**

$E(X_k) = 1(1/2) + 0(1/2) = 1/2$

$E(X^2_k) = 1(1/2) + 0(1/2) = 1/2$

$\sigma^2 = E(X^2_k) - [E(X_k)]^2 = 1/2 - (1/2)^2 = 1/4$

$E(S^2) = \frac{2-1}{2}\sigma^2 = (1/2)(1/4) = 1/8$