PROBABILITY THEORY
Lesson 15
Expectation Of A Random Variable

15.1- What is a the Expectation of a Random Variable?

15.1 - Problem 1:

$\mathbf{W}_1$: The event the first person selected is a woman.

$\mathbf{W}_2$: The event the second person selected is a woman.

$\mathbf{W}_3$: The event the third person selected is a woman.

Let X represent the number of women selected.

Case 1: {X = 0}: The event that no women were selected.

Step 1: $\{X = 0\} = \mathbf{W}_1'\cap\mathbf{W}_2'\cap\mathbf{W}_3'$

Step 2: $P\{X = 0\} = P(\mathbf{W}_1'\cap\mathbf{W}_2'\cap\mathbf{W}_3') = (\frac{5}{20})(\frac{4}{19})(\frac{3}{18}) = \frac{60}{6840}$

Case 2: {X = 1}: The event that only 1 women was selected.

Step 1: $\{X = 1\} = (\mathbf{W}_1\cap\mathbf{W}_2'\cap\mathbf{W}_3')\cup(\mathbf{W}_1'\cap\mathbf{W}_2\cap\mathbf{W}_3')\cup(\mathbf{W}_1'\cap\mathbf{W}_2'\cap\mathbf{W}_3) =$

Step 2: $P\{X = 2\} = P(\mathbf{W}_1\cap\mathbf{W}_2\cap\mathbf{W}_3') + P(\mathbf{W}_1\cap\mathbf{W}_2'\cap\mathbf{W}_3) + P(\mathbf{W}_1'\cap\mathbf{W}_2\cap\mathbf{W}_3)] =$

$3(\frac{15}{20})(\frac{5}{19})(\frac{4}{18}) = \frac{900}{6840}$

Case 3: {X = 2}: The event that 2 women were selected.

Step 1: $\{X = 2\} = (\mathbf{W}_1\cap\mathbf{W}_2\cap\mathbf{W}_3')\cup(\mathbf{W}_1'\cap\mathbf{W}_2\cap\mathbf{W}_3)\cup(\mathbf{W}_1\cap\mathbf{W}_2'\cap\mathbf{W}_3) =$

Step 2: $P\{X = 2\} = P(\mathbf{W}_1\cap\mathbf{W}_2\cap\mathbf{W}_3') + P(\mathbf{W}_1\cap\mathbf{W}_2'\cap\mathbf{W}_3) + P(\mathbf{W}_1'\cap\mathbf{W}_2\cap\mathbf{W}_3)] =$

Case 4: {X = 3}: The event that 3 women were selected.

Step 1: $\{X = 2\} = (\mathbf{W}_1\cap\mathbf{W}_2\cap\mathbf{W}_3)\cup(\mathbf{W}_1\cap\mathbf{W}_2\cap\mathbf{W}_3)\cup(\mathbf{W}_1\cap\mathbf{W}_2\cap\mathbf{W}_3) =$

Step 2: $P\{X = 2\} = P(\mathbf{W}_1\cap\mathbf{W}_2\cap\mathbf{W}_3) + P(\mathbf{W}_1\cap\mathbf{W}_2\cap\mathbf{W}_3) + P(\mathbf{W}_1\cap\mathbf{W}_2\cap\mathbf{W}_3)] =$

$(\frac{15}{20})(\frac{14}{19})(\frac{13}{18}) = \frac{2730}{6840}$

The distribution table is

X	**P{X = x}**	**xP{X = x}**
0	$\frac{60}{6840}$	$(0)\frac{60}{6840} = \frac{0}{6840}$
1	$\frac{900}{6840}$	$(1)\frac{900}{6840} = \frac{900}{6840}$
2	$\frac{3150}{6840}$	$(2)\frac{3150}{6840} = \frac{6300}{6840}$
3	$\frac{2730}{6840}$	$(3)\frac{2730}{6840} = \frac{8190}{6840}$
		E(X) = $\frac{15390}{6840} = 2.25$

15.1 - Problem 2:
The random variable X is the amount she wins/losses:

The sample space is **S** = {0,00,1,2,3,4,5,...,36}

{X = \$165 } = {0}
{X = \$345} = {00}
{X = - \$15} = {1,2,3,4,5,...,36}

P{X = \$165} = P{0} = 1/38
{X = \$345} = P{00} = 1/38
P{X = - \$15} = P{1,2,3,4,5,,,36} = 36/38

The distribution table is

X	**P{X = x}**	**xP{X = x}**
\$165	1/38	165/38
\$345	1/38	345/38
-\$15	36/38	-540/38
		E(X) = -30/38= \$ 0.79

15.1 - Problem 3:
The random variable is the number of selections X:

$\{X = 2\} = \mathbf{W_1} \cap \mathbf{W_2}$

$\{X = 3\} = (\mathbf{W_1}' \cap \mathbf{W_2} \cap \mathbf{W_3}) \cup (\mathbf{W_1} \cap \mathbf{W_2}' \cap \mathbf{W_3})$

$\{X = 4\} = (\mathbf{W_1}' \cap \mathbf{W_2}' \cap \mathbf{W_3} \cap \mathbf{W_4}) \cup (\mathbf{W_1}' \cap \mathbf{W_2} \cap \mathbf{W_3}' \cap \mathbf{W_4}) \cup (\mathbf{W_1} \cap \mathbf{W_2}' \cap \mathbf{W_3}' \cap \mathbf{W_4})$

Step 3: $P\{X = 2\} = P\{\mathbf{W_1} \cap \mathbf{W_2}\} = \left(\frac{2}{4}\right)\left(\frac{1}{3}\right) = \frac{2}{12}$

$P\{X = 3\} = P[(\mathbf{W_1}' \cap \mathbf{W_2} \cap \mathbf{W_3}) \cup (\mathbf{W_1} \cap \mathbf{W_2}' \cap \mathbf{W_3})] = P(\mathbf{W_1}' \cap \mathbf{W_2} \cap \mathbf{W_3}) + P(\mathbf{W_1} \cap \mathbf{W_2}' \cap \mathbf{W_3}) =$

$$\left(\frac{2}{4}\right)\left(\frac{2}{3}\right)\left(\frac{1}{2}\right) + \left(\frac{2}{4}\right)\left(\frac{2}{3}\right)\left(\frac{1}{2}\right) = \frac{4}{12}$$

P{X = 4} = 1 - P{X = 2} - P{X = 3} = 6/12

Step 4:

X = x	**P{X = x}**	**xP{X = x}**
2	2/12	4/12
3	4/12	12/12
4	6/12	24/12
	Total 1	**E(X)** = 40/12 ≈ 3.33 selections

15.1 - Problem 4:
The total number of hours she worked per week equals

T = (1)9 + (2)8 + (2)7 = 39.

The average number of hours she works per day is E(X) = 39/5 = 7.8 hours a day.

15.2 - The expectation of a sum of random variables.

15.2 - Problem 1:
Step 1:
Let X: the amount he wins at football.

Let Y: the amount he wins at baseball.

Let Z: the amount he wins at basketball.

Step 2:
E(X) = $100(0.60) - $110(0.40) = $16

E(Y) = $100(0.65) - $110(0.35) = $26.50

E(Z) = $100(0.70) - $110(0.30) = $37

Step 3: His average winning is

E(X + Y + Z) = $16 + $26.50 + $37 = $79.50

Supplementary Problems

1.
Let X equal the winnings for each king drawn.

Let Y equal the losses for each queen of hearts and diamonds drawn.

X = x	**P{X = x}**	**xP{X = x}**
$0 $100 $200	2256/2652 384/2652 12/2652	0 38400/2652 2400/2652
	1	E(X) = $40800/2652

Y = y	P{Y = y}	yP{Y = y}
$0 -$125 -$250	2450/2652 200/2652 2/2652	$0 -$25000/2652 -$500/2652
	1	E(Y) = $25500/2652

$E(X + Y) = \$40800/2652 - \$25500/2652 \approx \$5.77$

2.

Let X equal the amount won/lost on playing each individual game.
Let Y equal the amount won/lost on playing a two team parley.

X = x	**P{X = x}**	**xP{X = x}**
\$100	0.36	\$36
-\$ 5	0.48	-\$ 2.40
-\$110	0.16	-\$17.60
	1	E(X) = \$16.00

Y = y	**P{Y = y}**	**yP{Y = y}**
\$260	0.36	\$93.60
-\$100	0.64	-\$64.00
	1	E(Y) = \$29.60

The expected winning is $E(X + Y) = \$16 + 29.60 = \$45.60$.

3.

$P\{X = 0\} = P\{X \le 0\} = 0$

$$P\{X = 1\} = P\{X \le 1\} - P\{X \le 0\} = \frac{1^2}{25} - 0 = \frac{1}{25}$$

$$P\{X = 2\} = P\{X \le 2\} - P\{X \le 1\} = \frac{2^2}{25} - \frac{1^2}{25} = \frac{3}{25}$$

$$P\{X = 3\} = P\{X \le 3\} - P\{X \le 2\} = \frac{3^2}{25} - \frac{2^2}{25} = \frac{5}{25}$$

$$P\{X = 4\} = P\{X \le 4\} - P\{X \le 3\} = \frac{4^2}{25} - \frac{3^2}{25} = \frac{7}{25}$$

$$P\{X = 5\} = P\{X \le 5\} - P\{X \le 4\} = \frac{5^2}{25} - \frac{4^2}{25} = \frac{9}{25}$$

X	P{X = x}	xP{X = x}
0	0	0
1	1/25	1/25
2	3/25	6/25
3	5/25	15/25
4	7/25	28/25
5	9/25	45/25
	Total 1	**E(X)** = 95/25

4.

➤ a.

Let Z equal the total revenue.

{Z = \$900 + \$500 = \$1,400} = {X = \$900}∩{X = \$500}

P{Z = \$900 + \$500 = \$1,400} = P[{X = \$900}P{X = \$500} = (0.6)(0.7) = 0.42

{Z = \$900 + \$1100 = \$2,000} = [{X = \$900}∩{X = \$1100}]∪[{X = \$1500}∩{X = \$500}]

P{Z = \$900 + \$1100 = \$2,000} = P[{X = \$900}∩{X = \$1100}] + P[{X = \$1500}∩{X = \$500}] =

(0.6)(0.25) + (0.3)(0.7) = 0.15 + 0.21 = 0.36

{Z = \$900 + \$1500 = \$2,400} = {X = \$900}∩{X = \$1500}

P{Z = \$900 + \$1500 = \$2,400} = P[{X = \$900}P{X = \$1500} = (0.6)(0.05) = 0.03

{Z = \$2000 + \$500 = \$2,500} = {X = \$2000}∩{X = \$500}

P{Z = \$2000 + \$500 = \$2,500} = P{X = \$2000}P{X = \$500} = (0.10)(0.70) = 0.07

{Z = \$1500 + \$1100 = \$2,600} = {X = \$1500}∩{X = \$1100}

P{Z = \$1500 + \$1100 = \$2,500} = P{X = \$1500}P{X = \$1100} = (0.30)(0.25) = 0.075

{Z = \$1500 + \$1500 = \$3,000} = {X = \$1500}∩{X = \$1500}

P{Z = \$1500 + \$1500 = \$3,000} = P{X = \$1500}P{X = \$1500} = (0.30)(0.05) = 0.015

{Z = \$2000 + \$1100 = \$3,100} = {X = \$2000}∩{X = \$1100}

P{Z = \$2000 + \$1100 = \$3,100} = P{X = \$2000}P{X = \$1100} = (0.10)(0.25) = 0.025

{Z = \$2000 + \$1500 = \$3,500} = {X = \$2000}∩{X = \$1500}

P{Z = \$2000 + \$1500 = \$3,500} = P{X = \$2000}P{X = \$1500} = (0.10)(0.05) = 0.005

Z = X + Y = z	P{Z = z}	zP{Z = z}
\$1400	0.42	\$588
\$2000	0.36	\$720
\$2400	0.03	\$72
\$2500	0.07	\$175
\$2600	0.075	\$195
\$3000	0.015	\$ 45
\$3100	0.025	\$77.5
\$3500	0.005	\$17.5
	Total = 1	**E(Z)** = \$1,890

➤ b.

E(Z) = \$1,890 = E(X) + E(Y) \$1190 + \$700 = \$1890 = E(X + Y)

5.

➤ a.

$\mathbf{D_1}$: The event that a the first card drawn is a diamond.
$\mathbf{D_2}$: The event that a the second card drawn is a diamond.

Let X equal the number of diamonds drawn.

$\{X = 0\} = \mathbf{D_1}' \cap \mathbf{D_2}'$

$P\{X = 0\} = P(\mathbf{D_1}' \cap \mathbf{D_2}') = P(\mathbf{D_1}')P(\mathbf{D_2}' | \mathbf{D_1}') = (\frac{39}{52})(\frac{38}{51}) = \frac{1482}{2652}$

$\{X = 1\} = (\mathbf{D_1} \cap \mathbf{D_2}') \cup (\mathbf{D_1}' \cap \mathbf{D_2})$

$P\{X=1\} = P(\mathbf{D_1} \cap \mathbf{D_2}') + P(\mathbf{D_1}' \cap \mathbf{D_2}) = P(\mathbf{D_1})P(\mathbf{D_2}' | \mathbf{D_1}) + P(\mathbf{D_1}')P(\mathbf{D_2} | \mathbf{D_1}') =$

$(\frac{13}{52})(\frac{39}{51}) + (\frac{39}{52})(\frac{13}{51}) = \frac{1014}{2652}$

$P\{X = 2\} = P(\mathbf{D_1} \cap \mathbf{D_2}) = P(\mathbf{D_1})P(\mathbf{D_2} | \mathbf{D_1}) = (\frac{13}{52})(\frac{12}{51}) = \frac{156}{2652}$

X	P{X = x}
0	$\frac{1482}{2652}$
1	$\frac{1014}{2652}$
2	$\frac{156}{2652}$
	Total 1

➤ b.

X	P{X = x}	xP{X = x}
0	$\frac{1482}{2652}$	0
1	$\frac{1014}{2652}$	$\frac{1014}{2652}$
2	$\frac{156}{2652}$	$\frac{312}{2652}$
	Total 1	**E(X)** $= \frac{221}{442} = 0.5$

6.

➤ a.

Let p equal the probability that Mr. Smith will win the single game.

Since this is a fair game,

$E(X) = (\$100)P - (\$110)(1 - P) = \$210p - \$110 = 0.$

Solving for p we have $p = \frac{110}{210} \approx 0.523$

➤ b.

Two team parley:

Step 1: p equals the probability that each game of the two team parley will win. Therefore, the chance of winning the parley is $pp = p^2$.

Step 2: Since this is a fair game and the payoff for winning is \$260,

$E(X) = (\$260)p^2 - (\$100)(1 - p^2) = \$360p^2 - \$100 = 0.$

$p^2 = \frac{100}{360}$

Solving for p we have $p = \sqrt{\frac{100}{360}} \approx 0.527$

<u>Three team parley:</u>

Step 1: p equals the probability that each game of the three team parley will win. Therefore, the chance of winning the parley is $ppp = p^3$.

Step 2: Since this is a fair game and the payoff for winning is $600,

$E(X) = (\$600)p^3 - (\$100)(1 - p^3) = \$700p^3 - \$100 = 0.$

$p^3 = \frac{100}{700}$

Solving for p we have $p = \sqrt[3]{\frac{100}{700}} \approx 0.523$

<u>Four team parley:</u>

Step 1: p equals the probability that each game of the four team parley will win. Therefore, the chance of winning the parley is $pppp = p^4$.

Step 2: Since this is a fair game and the payoff for winning is $600,

$E(X) = (\$1000)p^4 - (\$100)(1 - p^4) = \$1100p^4 - \$100 = 0.$

$p^4 = \frac{100}{1100}$

Solving for p we have $p = \sqrt[4]{\frac{100}{1100}} \approx 0.549$

<u>Five team parley:</u>

Step 1: p equals the probability that each game of the four team parley will win. Therefore, the chance of winning the parley is $ppppp = p^5$.

Step 2: Since this is a fair game and the payoff for winning is $600,

$E(X) = (\$2000)p^5 - (\$100)(1 - p^5) = \$2100p^5 - \$100 = 0.$

$p^5 = \frac{100}{2100}$

Solving for p we have $p = \sqrt[5]{\frac{100}{2100}} \approx 0.544$

The completed table is

Number of teams in the parley	Winning	P
2	\$ 260	0.527
3	\$ 600	0.523
4	\$1,000	0.549
5	\$2,000	0.544

7.

Step 1: X_k (k = 1 ...,65): his winnings on football.

Y_k (k = 1 ...,125): his winnings on baseball.

Z_k (k = 1 ...,200): his winnings on basketball.

Step 2: $E(X_k) = \$100(0.60) - \$110(0.40) = \$16$

$E(Y_k) = \$100(0.65) - \$110(0.35) = \$26.50$

$E(Z_k) = \$100(0.70) - \$110(0.30) = \$37$

Step 3: $X = X_1 + X_2 + ...+ X_{65}$: his total winnings on football.

$Y = Y_1 + Y_2 + ...+ Y_{125}$: his total winnings on baseball.

$Z = Z_1 + Z_2 + ...+ Z_{200}$: his total winnings on basketball.

Step 4: $E(X + Y + Z)$: his average winnings.

$E(X + Y + Z) = E(X) + E(Y) + E(Z) = E(X_1 + X_2 + ...+ X_{65}) + E(Y_1 + Y_2 + ...+ Y_{125}) +$

$E(Z_1 + Z_2 + ...+ Z_{200}) =$

$E(X_1) + E(X_2) + ... + E(X_{65}) + E(Y_1) + E(Y_2) + ... + E(Y_{125}) + E(Z_1) + E(Z_2) + ... + E(Z_{200}) =$

65($16) + 125($26.50) + 200($37) = $11,752.50

8.
Define the conditional expectation of X given Y as

$$E(X|Y = y_k) = x_1P(X = x_1|Y = y_k) + x_2P(X = x_2|Y = y_k) + ... + x_nP(X = x_n|Y = y_k)$$

For 10 shots, let X equal the maximum number of misses between two hits and Y the number of hits. Assume Y = 2.

Let **E** be the event that in 10 shots, she hits the target only twice. The following number of possibilities can happen:

Zero misses between hits. This can happen in 9 different ways.
One miss between hits. This can happen in 8 different ways.
Two misses between hits. This can happen in 7 different ways.
Three misses between hits. This can happen in 6 different ways.
Four misses between hits. This can happen in 5 different ways.
Five misses between hits. This can happen in 4 different ways.
Six misses between hits. This can happen in 3 different ways.
Seven misses between hits. This can happen in 2 different ways.
Eight misses between hits. This can happen in 1 different ways.

#**E** = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45.

$E(X|Y = 2) = 0P(X = 0|Y = 2) + 1P(X = 1|Y = 2) + 2P(X = 2|Y = 2) + 3P(X = 3|Y = 2) +$

$4P(X = 4|Y = 2) + 5P(X = 5|Y = 2) + 6P(X = 6|Y = 2) + 7P(X = 7|Y = 2) + 8P(X = 8|Y = 2) +$

$9P(X = 9|Y = 2) =$

$$0(\frac{9}{45}) + 1(\frac{8}{45}) + 2(\frac{7}{45}) + 3(\frac{6}{45}) + 4(\frac{5}{45}) + 5(\frac{4}{45}) + 6(\frac{3}{45}) + 7(\frac{2}{45}) + 8(\frac{1}{45}) =$$

$$\frac{8 + 14 + 18 + 20 + 20 + 18 + 14 + 8}{45} \approx 1.78$$

9.
Let $\mu = E(X)$.

$E(X - \mu)^2 = E(X^2 - 2X\mu + \mu^2) = E(X^2) - E(2X\mu) + E(\mu^2) = E(X^2) - 2E(\mu X) + E(\mu^2) =$

$E(X - \mu)^2 = E(X^2) - 2\mu E(X) + \mu^2 = E(X^2) - 2\mu^2 + \mu^2$

$E(X - \mu)^2 = E(X^2) - \mu^2$

$E(X - \mu)^2 + \mu^2 = E(X^2)$

SInce each term is non-negative,

$\mu^2 \le E(X^2)$

$[E(X)]^2 \le E(X^2)$

10.
$E(XY) = x_1y_1P[(X = x_1)\cap(Y = y_1)] + x_1y_2P[(X = x_1)\cap(Y = y_2)] + ... +$

$x_1y_nP[(X = x_1)\cap(Y = y_n)]) + x_2y_1P[(X = x_2)\cap(Y = y_1)] +$

$x_2y_2P[(X = x_2)\cap(Y = y_2)] + ...+ x_2y_nP[(X = x_2)\cap(Y = y_n)] + ... +$

$x_my_1P[(X = x_m)\cap(Y = y_1)]) + x_my_2P[(X = x_m)\cap(Y = y_2)] +$

$x_my_3P[(X = x_m)\cap(Y = y_3)] + ...+ x_my_nP[(X = x_m)\cap(Y = y_n)] =$

$x_1y_1P(X = x_1)P(Y = y_1)+ x_1y_2P(X = x_1)P(Y = y_2) + ... +$

$x_1y_nP(X = x_1)P(Y = y_n)$~+~$x_2y_1P(X = x_2)P(Y = y_1) +$

$x_2y_2P(X = x_2)P(Y = y_2) + ...+ x_2y_nP(X = x_2)P(Y = y_n) + ... +$

$x_my_1P(X = x_m)P(Y = y_1) + x_my_2P(X = x_m)P(Y = y_2) +$

$x_my_3P(X = x_m)P(Y = y_3) + ...+ x_my_nP(X = x_m)P(Y = y_n) =$

$x_1P(X = x_1)\{y_1P(Y = y_1) + y_2P(Y = y_2) + y_nP(Y = y_n)\} +$

$x_2P(X = x_2)\{y_1P(Y = y_1) + y_2P(Y = y_2) + ...+ y_nP(Y = y_n)\} +$

$x_mP(X = x_m)\{y_1P(Y = y_1) + y_2P(Y = y_2) + ...+ y_nP(Y = y_n)\} =$

$x_1P(X = x_1)E(Y) + x_2P(X = x_2)E(Y) + ... + x_mP(X = x_m)E(Y) =$

$[x_1P(X = x_1)+ x_2P(X = x_2) + x_mP(X = x_m)]E(Y) = E(X)E(Y).$

11.
Let X_k be the number of diamonds on the kth drawing (k = 1,2,..., 10).

Therefore, $P\{ X_k = 0\} = 39/52$ and $P\{ X_k = 1\} = 13/52$.

$E(X_k) = 0\ P\{ X_k = 0\} + 1\ P\{ X_k = 1\} = 0(39/52) + 1(13/52) = 13/52$

$X = X_1 + X_2 + ... + X_{10}$, the number of diamonds.

$E(X) = EX_1) + E(X_2) + ... + E(X_{10}) = 10(13/52) = 130/52 = 2.5$

12.
Case 1: $c \neq 0$

Step 1: $E(X) = x_1 P\{X = x_1\} + x_2 P\{X = x_2\} + ... + x_n P\{X = x_n\}$

Step 2: $P\{X = x_k\} = P\{cX = cx_k\}$, $k = 1,2,3,...,n$.

Step 3: $E(cX) = cx_1 P\{X = cx_1\} + cx_2 P\{X = cx_2\} + ... + cx_n P\{X = cx_n\}$

$c[x_1 P\{X = cx_1\} + x_2 P\{X = cx_2\} + ... + x_n P\{X = cx_n\}] = c[x_1 P\{X = x_1\} + x_2 P\{X = x_2\} + ... +$

$x_n P\{X = x_n\}] = E(X)$

Case 3: $c = 0$.

$E(0X) = E(0) = 0E(X) = 0$

13.
Step 1: Since t is a constant (see problem 12), we can write:

$E[(tX + Y)^2] = E(t^2 X^2 + 2XYt + Y^2) = E(X^2)t^2 + 2E(XY)t + E(Y^2) \geq 0$

Step 2: Since $E(X^2)t^2 + 2E(XY)t + E(Y^2)$ is a quadratic polynomial for the variable t, and is non-negative, from the discriminate of the quadratic formula, we have

$[2E(XY)]^2 - 4E(X^2)E(Y^2) \leq 0$.

Step 3: This gives us $4[E(XY)]^2 \leq 4E(X^2)E(Y^2)$

Dividing out the 4, give us $[E(XY)]^2 \leq E(X^2)E(Y^2)$.

14.
$\mathbf{F_k}$: The event that a five occurs on the kth toss (k = 1,2,3,4).

The event **E** can occur in the following ways:

$(\mathbf{F_1} \cap \mathbf{F_2})$: The event that only two tosses happened.

$(\mathbf{F_1}' \cap \mathbf{F_2} \cap \mathbf{F_3}) \cup (\mathbf{F_1} \cap \mathbf{F_2}' \cap \mathbf{F_3})$: The event that only three tosses happened.

$[(\mathbf{F}_1 \cap \mathbf{F}_2) \cup (\mathbf{F}_1' \cap \mathbf{F}_2 \cap \mathbf{F}_3) \cup (\mathbf{F}_1 \cap \mathbf{F}_2' \cap \mathbf{F}_3)]'$: The event that four tosses happened.

Let X equal the number of possible tosses.

$P\{X = 2\} = P\{\mathbf{F}_1 \cap \mathbf{F}_2\} = (1/6)(1/6) = 1/36$

$P\{X = 3\} = P[(\mathbf{F}_1' \cap \mathbf{F}_2 \cap \mathbf{F}_3) \cup (\mathbf{F}_1 \cap \mathbf{F}_2' \cap \mathbf{F}_3)] = P(\mathbf{F}_1' \cap \mathbf{F}_2 \cap \mathbf{F}_3) + P(\mathbf{F}_1 \cap \mathbf{F}_2' \cap \mathbf{F}_3) =$

$(5/6)(1/6)(1/6) + (5/6)(1/6)(1/6) = 5/216$

$P\{X = 4\} = P[(\mathbf{F}_1 \cap \mathbf{F}_2) \cup (\mathbf{F}_1' \cap \mathbf{F}_2 \cap \mathbf{F}_3) \cup (\mathbf{F}_1 \cap \mathbf{F}_2' \cap \mathbf{F}_3)]' = 1 - 1/36 - 5/216 = 205/216$

X	**P{X = x}**	**xP{X = x}**
2	1/36	2/36
3	5/216	15/216
4	205/216	820/216
		E(X) = 847/216 ≈ 3.92 tosses

15.

$$E(\overline{X}) = E\left(\frac{X_1}{n} + \frac{X_2}{n} + \ldots + \frac{X_n}{n}\right) = E\left(\frac{X_1}{n}\right) + E\left(\frac{X_2}{n}\right) + \ldots + E\left(\frac{X_n}{n}\right) =$$

$$\frac{E(X_1)}{n} + \frac{E(X_2)}{n} + \ldots \frac{E(X_n)}{n} = n\frac{\mu}{n} = \mu$$

16.

➤ a.

$P\{X \le k\}; k = 1,2,\ldots,6$

$\mathbf{M}_i$: The event that the i th toss (i = 1,2,3) resulted in a number less than or equal k.

$P\{\mathbf{M}_i\} = k/6$, since each number of the die as an equal chance of occuring.

$\{X \le k\} = \mathbf{M}_1 \cap \mathbf{M}_2 \cap \mathbf{M}_3$

Since the event $\mathbf{M}_i$ are mutually independent, we have

$P\{X \le k\} = P\{\mathbf{M}_1 \cap \mathbf{M}_2 \cap \mathbf{M}_3\} = P(\mathbf{M}_1)P(\mathbf{M}_2)P(\mathbf{M}_3) = (k/6)^3 = k^3/216$

➤ b.

$P\{X = k\} = P\{X \le k\} - P\{X \le k\text{-}1\} = [k^3 - (k - 1)^3]216^{-1}$

➤ c.

$E(X) = 1P\{X = 1\} + 2P\{X = 2\} + 3P\{X = 3\} + 4P\{X = 4\} + 5P\{X = 5\} + 6P\{X = 6\} =$

$[1 + 2(8 - 1) + 3(27 - 8) + 4(64 - 27) + 5(125 - 64) + 6(216 - 125)]216^{-1} =$

$[1 + 2(7) + 3(19) + 4(37) + 5(61) + 6(91)]216^{-1} = [1 + 14 + 57 + 148 + 305 + 546]216^{-1} =$

$1071/216 \approx 4.96$

17.

➤ a.

$P\{X_1 = 0\,; X_2 = 0\} = P\{(X_1 = 0)\cap(X_2 = 0)\} = (39/52)(38/51) = 1482/2652$

$P\{X_1 = 0\,; X_2 = 1\} = P\{(X_1 = 0)\cap(X_2 = 1)\} = (39/52)(13/51) = 507/2652$

$P\{X_1 = 0\,; X_2 = 2\} = P\{(X_1 = 0)\cap(X_2 = 2)\} = (39/52)(0) = 0$

$P\{X_1 = 1\,; X_2 = 0\} = P\{(X_1 = 1)\cap(X_2 = 0)\} = (13/52)(0) = 0$

$P\{X_1 = 1\,; X_2 = 1\} = P\{(X_1 = 1)\cap(X_2 = 1)\} = (13/52)(39/51) = 507/2652$

$P\{X_1 = 1\,; X_2 = 2\} = P\{(X_1 = 1)\cap(X_2 = 2)\} = (13/52)(12/51) = 156/2652$

$P\{X_1 = 0\,; X_3 = 0\} = P\{(X_1 = 0)\cap(X_2 = 0)\cap(X_3 = 0)\} = (39/52)(38/51)(37/50) = 54834/132600$

$\{X_1 = 0\,; X_3 = 1\} = [(X_1 = 0)\cap(X_2 = 0)\cap(X_3 = 1)]\cup[(X_1 = 0)\cap(X_2 = 1)\cap(X_3 = 1)]$

$P\{X_1 = 0\,; X_3 = 1\} = P[(X_1 = 0)\cap(X_2 = 0)\cap(X_3 = 1)] + P[(X_1 = 0)\cap(X_2 = 1)\cap(X_3 = 1)] =$

$(39/52)(38/51)(13/550) + (39/52)(13/51)(38/50) = 38532/132600$

$P\{X_1 = 0\,; X_3 = 2\} = P[(X_1 = 0)\cap(X_2 = 1)\cap(X_3 = 1)] = (39/52)(13/51)(12/50) = 6084/132600$

$P\{X_1 = 0\,; X_3 = 3\} = P[(X_1 = 0\}\cap(X_3 = 3)] = (39/52)(0) = 0$

$P\{X_1 = 1\,; X_3 = 0\} = P[(X_1 = 1\}\cap(X_3 = 0)] = (13/52)(0) = 0$

$P\{X_1 = 1\,; X_3 = 1\} = P[(X_1 = 1\}\cap(X_2 = 1)\cap(X_3 = 1)] = (13/52)(39/51)(38/50) = 19266/132600$

$P\{X_1 = 1\,; X_3 = 2\} = P[(X_1 = 1)\cap(X_2 = 1)\cap(X_3 = 2)] + P[(X_1 = 1)\cap(X_2 = 2)\cap(X_3 = 2)] =$

$(13/52)(39/51)(12/50) + (13/52)(12/51)(39/50) = 12168/132600$

$P\{X_1 = 1; X_3 = 3\} = P[(X_1 = 1)\cap(X_2 = 2)\cap(X_3 = 3)] = (13/52)(12/51)(11/50) = 1716/132600$

$P\{X_2 = 0 ; X_3 = 0\} = P[(X_1 = 0)\cap(X_2 = 0)\cap(X_3 = 0)] = (39/52)(38/51)(37/50) = 54834/132600$

$P\{X_2 = 0 ; X_3 = 1\} = P[(X_1 = 0)\cap(X_2 = 0)\cap(X_3 = 1)] = (39/52)(38/51)(13/50) = 19266/132600$

$P\{X_2 = 0 ; X_3 = 3\} = P[(X_1 = 0)\cap(X_2 = 0)\cap(X_3 = 3)] = (39/52)(38/51)(0) = 0$

$P\{X_2 = 1; X_3 = 0\} = P[(X = 0)\cap(X_2 = 1)\cap(X_3 = 0)] + P[(X_1 = 1)\cap(X_2 = 1)\cap(X_3 = 0)] =$

$(39/52)(13/51)(0) + (13/52)(39/51)(0) = 0$

$P\{X_2 = 1; X_3 = 1\} = P[(X_1 = 0)\cap(X_2 = 1)\cap(X_3 = 1)] + P[(X_1 = 1)\cap(X_2 = 1)\cap(X_3 = 1)] =$

$(39/52)(13/51)(38/50) + (13/52)(39/51)(38/50) = 19266/132600$

$P\{X_2 = 1; X_3 = 3\} = P[(X_1 = 1)\cap(X_2 = 2)\cap(X_3 = 3)] = (13/52)(12/51)(11/50) = 1716/132600$

$P\{X_2 = 2 ; X_3 = 2\} = P[(X_1 = 1)\cap(X_2 = 2)\cap(X_3 = 2)] = (13/52)(12/51)(39/50) = 6084/132600$

$P\{X_2 = 2 ; X_3 = 3\} = P[(X_1 = 1)\cap(X_2 = 2)\cap(X_3 = 3)] = (13/52)(12/51)(11/50) = 1716/132600$

$P\{X_1 = 0 ; X_2 = 0 ; X_3 = 0\} = P[(X_1 = 0)\cap(X_2 = 0)\cap(X_3 = 0)] = (39/52)(38/51)(37/50) =$

$54834/132600$

$P\{X_1 = 0 ; X_2 = 1; X_3 = 0\} = P[(X_1 = 0)\cap(X_2 = 1)\cap(X_3 = 0)] = (39/52)(13/51)(0) = 0$

$P\{X_1 = 0 ; X_2 = 1; X_3 = 1\} = P[(X_1 = 0)\cap(X_2 = 1)\cap(X_3 = 1)] = (39/52)(13/51)(38/50) =$

$19266/132600$

$P\{X_1 = 0 ; X_2 = 1; X_3 = 2\} = P[(X_1 = 0)\cap(X_2 = 1)\cap(X_3 = 2)] = (39/52)(13/51)(12/50) =$

$6084/132600$

$P\{X_1 = 0 ; X_2 = 1; X_3 = 3\} = P[(X_1 = 0)\cap(X_2 = 1)\cap(X_3 = 3)] = (39/52)(13/51)(0) = 0$

$P\{X_1 = 0 ; X_2 = 2; X_3 = 0\} = P[(X_1 = 0)\cap(X_2 = 2)\cap(X_3 = 0)] = (39/52)(0)(0) = 0$

$P\{X_1 = 0 ; X_2 = 2; X_3 = 1\} = P[(X_1 = 0)\cap(X_2 = 2)\cap(X_3 = 1)] = (39/52)(0)(0) = 0$

$P\{X_1 = 0 ; X_2 = 2; X_3 = 2\} = P[(X_1 = 0) \cap (X_2 = 2) \cap (X_3 = 2)] = (39/52)(0)(0) = 0$

$P\{X_1 = 0 ; X_2 = 2; X_3 = 3\} = P[(X_1 = 0) \cap (X_2 = 2) \cap (X_3 = 3)] = (39/52)(0)(0) = 0$

$P\{X_1 = 1 ; X_2 = 0; X_3 = 0\} = P[(X_1 = 1) \cap (X_2 = 0) \cap (X_3 = 0)] = (13/52)(0)(0) = 0$

$P\{X_1 = 1 ; X_2 = 0; X_3 = 1\} = P[(X_1 = 1) \cap (X_2 = 0) \cap (X_3 = 1)] = (13/52)(0)(0) = 0$

$P\{X_1 = 1 ; X_2 = 0; X_3 = 2\} = P[(X_1 = 1) \cap (X_2 = 0) \cap (X_3 = 2)] = (13/52)(0)(0) = 0$

$P\{X_1 = 1 ; X_2 = 0; X_3 = 3\} = P[(X_1 = 1) \cap (X_2 = 0) \cap (X_3 = 3)] = (13/52)(0)(0) = 0$

$P\{X_1 = 1 ; X_2 = 1; X_3 = 0\} = P[(X_1 = 1) \cap (X_2 = 1) \cap (X_3 = 0)] = (13/52)(39/51)(0) = 0$

$P\{X_1 = 1 ; X_2 = 1; X_3 = 1\} = P[(X_1 = 1) \cap (X_2 = 1) \cap (X_3 = 1)] = (13/52)(39/51)(38/50) =$

$19266/132600$

$P\{X_1 = 1 ; X_2 = 1; X_3 = 2\} = P[(X_1 = 1) \cap (X_2 = 1) \cap (X_3 = 2)] = (13/52)(39/51)(12/50) =$

$6084/132600$

$P\{X_1 = 1 ; X_2 = 1; X_3 = 3\} = P[(X_1 = 1) \cap (X_2 = 1) \cap (X_3 = 3)] = (13/52)(39/51)(0) = 0$

$P\{X_1 = 1 ; X_2 = 2; X_3 = 0\} = P[(X_1 = 1) \cap (X_2 = 2) \cap (X_3 = 0)] = (13/52)(12/51)(0) = 0$

$P\{X_1 = 1 ; X_2 = 2; X_3 = 1\} = P[(X_1 = 1) \cap (X_2 = 2) \cap (X_3 = 1)] = (13/52)(12/51)(0) = 0$

$P\{X_1 = 1 ; X_2 = 2; X_3 = 2\} = P[(X_1 = 1) \cap (X_2 = 2) \cap (X_3 = 2)] = (13/52)(12/51)(39/50) =$

$6084/132600$

$P\{X_1 = 1 ; X_2 = 2; X_3 = 3\} = P[(X_1 = 1) \cap (X_2 = 2) \cap (X_3 = 3)] = (13/52)(12/51)(11) =$

$1715/132600$

➤ b.

$P\{X_1 = 1 ; X_2 = 0\} = P\{(X_1 = 1) \cap (X_2 = 0)\} = (13/52)(0) = 0$

$P(X_1 = 1)P(X_2 = 0) = P(X_1 = 1)P[(X_1 = 0) \cap (X_2 = 0)] = (13/52)(39/52)(38/51) \neq 0$

18.

➤ a.

Step 1: Define $p(x_i,y_j) = P\{X + Y = x_i + y_j\} = P\{(X = x_i)\cap(Y = y_j)\}$

Step 2: $E(X + Y) = (x_1 + y_1)P\{X + Y = x_1 + y_1\} + (x_1 + y_2)P\{X + Y = x_1 + y_2\} +$

$(x_2 + y_1)P\{X + Y = x_2 + y_1\} + (x_2 + y_2)P\{X + Y = x_2 + y_2\} =$

$(x_1 + y_1)p(x_1, y_1) + (x_1 + y_2)p(x_1,y_2) + (x_2 + y_1)p(x_2, y_1) + (x_2 + y_2)p(x_2,y_2) =$

Step 3:

$E(X + Y) = x_1\{p(x_1,y_1) + p(x_1, y_2)\} + x_2\{p(x_2 ,y_1) + p(x_2, y_2)\} +$

$y_1\{p(x_1,y_1) + p(x_2, y_1)) + y_2\{p(x_1 ,y_2) + p(x_2, y_2)\} =$

$x_1P(X = x_1) + x_2P(X = x_2) + y_1P(Y = y_1) + y_2P(Y = y_2) = E(X) + E(Y)$

➤ b.

We proceed as in a.

$E(X + Y) =$

$x_1\{p(x_1,y_1) + p(x_1, y_2) + ... p(x_1,y_m)\} + ... + x_n\{p(x_n ,y_1) + p(x_n, y_2) + ... + p(x_n,y_m)\} +$

$y_1\{p(x_1,y_1) + p(x_2, y_1) + ... p(x_n,y_1)\} + ... + y_m\{p(x_1 ,y_m) + p(x_2, y_m) + ... + p(x_n,y_m)\} =$

$x_1P(X = x_1) + x_2P(X = x_2) + ... + x_nP(X = x_n) + y_1P(Y = y_1) + y_2P(Y = y_2) + ... + y_mP(Y = y_m) =$

$E(X + Y)$

➤ c.

Assume a and c are constants.

Step 1 $E(aX) = ax_1P(X = x_1) + ax_2P(X = x_2) + ... + ax_nP(X = x_n) =$

$E(aX) = a[x_1P(X = x_1) + x_2P(X = x_2) + ... + x_nP(X = x_n)] = aE(X)$

Similarly, $E(bY) = bE(Y)$.

From b., we have

$E(aX + bY) = E(aX) + E(by) = aE(X) + bE(Y)$.

➤ d.
Define $Z = X_1 + X_2 + ... + X_{n-1}$.

Therefore,

$E(X_1 + ... + X_n) = E(Z + X_n) = E(Z) + E(X_n)$

Continuing to break down Z in a similiar manner we have

$E(X_1 + ... + X_n) = E(X_1) + ... + E(X_n)$

19.
➤a.
$E(X = x|Z = z) = x_1P(X = x_1|Z = z) + x_2P(X = x_2|Z = z) + ... + x_nP(X = x_n|Z = z)$

First we shall show $E(X + Y|Z = z) = E(X|Z = z) + E(Y|Z = z)$.

Step 1: Define $p_z(x_i,y_j) = P\{X + Y = x_i + y_j |Z = z\} = P\{(X = x_i)\cap(Y = y_j)| Z = z\}$

Step 2: $E(X + Y) = (x_1 + y_1)P\{X + Y = x_1 + y_1|Z = z\} + (x_1 + y_2)P\{X + Y = x_1 + y_2| Z = z\} +$

$(x_2 + y_1)P\{X + Y = x_2 + y_1| Z = z\} + (x_2 + y_2)P\{X + Y = x_2 + y_2| Z = z\} =$

$(x_1 + y_1)p_z(x_1, y_1) + (x_1 + y_2)p_z(x_1,y_2) + (x_2 + y_1)p_z(x_2, y_1) + (x_2 + y_2)p_z(x_2,y_2) =$

Step 3: $E(X + Y|Z = z) = x_1\{p_z(x_1,y_1) + p_z(x_1, y_2)\} + x_2\{p_z(x_2 ,y_1) + p_z(x_2, y_2)\} +$

$y_1\{p_z(x_1,y_1) + p_z(x_2, y_1)) + y_2\{p_z(x_1 ,y_2) + p_z(x_2, y_2)\} =$

$x_1P(X = x_1|Z = z) + x_2P(X = x_2|Z = z) + y_1P(Y = y_1|Z = z) + y_2P(Y = y_2|Z = z) =$
$E(X|Z = z) + E(Y|Z = z)$

➤b.
Proceeding as in a. we have

$E(X + Y|Z = z) =$

$x_1\{p_z(x_1,y_1) + p_z(x_1, y_2) + ... p_z(x_1,y_m)\} + ... + x_n\{p_z(x_n ,y_1) + p_z(x_n, y_2) + ... + p_z(x_n,y_m)\} +$

$y_1\{p_z(x_1,y_1) + p_z(x_2, y_1) + ... p_z(x_n,y_1)\} + ... + y_m\{p_z(x_1 ,y_m) + p_z(x_2, y_m) + ... + p_z(x_n,y_m)\} =$

$x_1P(X = x_1|Z = z) + x_2P(X = x_2|Z = z) + ... + x_nP(X = x_n|Z = z) + y_1P(Y = y_1|Z = z) +$

$y_2P(Y = y_2|Z = z) + ... + y_mP(Y = y_m|Z = z) = E(X + Y|Z = z)$

➤c.

$E(aX|Z = z) =$

$E(X = ax|Z = z) = ax_1P(X = {}_{X1}|Z = z) + ax_2P(X = {}_{X2}|Z = z) + ... + ax_nP(X = {}_{Xn}|Z = z) =$

$a[x_1P(X = {}_{X1}|Z = z) + x_2P(X = {}_{X2}|Z = z) + ... + x_nP(X = {}_{Xn}|Z = z)] = aE(X|Z = z)$

Applying b., we have

$E(aX + bY|Z = z) = E(aX|Z = z) + E(bY|Z = z) = aE(X|Z = z) + bE(Y|Z = z)$

➤d.

Let X = c, where c is a constant.

$(X = c) =$ **S**

$E(X = c|Z = z) = cP(X = c|Z = z) = cP[(X = c)\cap(Z = z)]/P(Z = z) = cP(Z = z)/P(Z = z) = c$

➤e.

$E(X = x|Y = y) = x_1P(X = {}_{X1}|Y = y) + x_2P(X = {}_{X2}|Y = y) + ... + x_nP(X = {}_{Xn}|Y = y) =$

$x_1P[(X = {}_{X1})\cap(Y = y)]/P(Y = y) + x_2P[(X = {}_{X2})\cap(Y = y)]/P(Y = y) + ... +$

$x_nP[(X = {}_{Xn})\cap(Y = y)]/P(Y = y) =$

$x_1P(X = {}_{X1})P(Y = y)/P(Y = y) + x_2P(X = {}_{X2})\cap P(Y = y)/P(Y = y) + ... +$

$x_nP(X = {}_{Xn})P(Y = y)/P(Y = y) = x_1P(X = {}_{X1}) + x_2P(X = {}_{X2}) + ... + x_nP[(X = {}_{Xn}) = E(X)$

20.

$E[E(X | Y)] = E(X | Y= y_1)P(Y = y_1) + E(X | Y= y_2)P(Y = y_2) + ... + E(X | Y= y_m)P(Y = y_m)$

$E(X | Y = y_k) = x_1P(X = x_1 | Y = y_k) + x_2P(X = x_2 | Y = y_k) + ... + x_nP(X = x_n | Y = y_k)$

$E(X | Y = y_k)P(Y = y_k) = x_1P(X = x_1 \cap Y = y_k) + x_2P(X = x_2 \cap Y = y_k) + ... + x_nP(X = x_n \cap Y = y_k)$

$E[E(X | Y)] =$

$x_1P(X = x_1 \cap Y = y_1) + x_2P(X = x_2 \cap Y = y_1) + ... + x_nP(X = x_n \cap Y = y_1) +$

$x_1P(X = x_1 \cap Y = y_2) + x_2P(X = x_2 \cap Y = y_2) + ... + x_nP(X = x_n \cap Y = y_2) + ... +$

$x_1P(X = x_1 \cap Y = y_m) + x_2P(X = x_2 \cap Y = y_m) + ... + x_nP(X = x_n \cap Y = y_m)$

Adding the above columns we have

$x_1P(X = x_1) + x_2P(X = x_2) + ... + x_nP(X = x_n) = E(X)$.

21.

➤a.

$\mu = E(X_k) = 1P(Xk = 1) + 0P(X_k = 0) = 1p = p$

➤b.

$E(S \mid N = k) = E(X_1 + X_2 + ... X_k \mid N = k) = E(X_1 \mid N = k) + E(X_2 \mid N = k) + ... + E(X_k \mid N = k)$

Assume $j \le k$, it follows

$E(X_j \mid N = k) = 1P(X_j = 1 \mid N = k) + 0P(X_j = 0 \mid N = k) = 1p = p = \mu$

$E(S \mid N = k) = E(X_1 \mid N = k) + E(X_2 \mid N = k) + ... + E(X_k \mid N = k) = \mu k$

Therefore, $E(S \mid N) = \mu N$.

➤c.

From problem 20, we have $E[E(S \mid N)] = E(S) = E[\mu N] = \mu E(N)$

22.

➤a.

Since X = c, a constant, $E(X = c \mid Y = y) = cP\{X = c \mid Y = y\} = c(1) = c$.

➤b.

$E(X \mid X = x_k) = x_1P\{X = x_1 \mid X = x_k\} + ... + x_kP\{X = x_k \mid X = x_k\} + ... + x_nP\{X = x_n \mid X = x_k\} =$

$x_10 + ... + x_{k-1}0 + x_k P\{X = x_k \mid X = x_k\} + x_{k+1}0 ... + xn0 = x_k1 = x_k$

➤c.

Assume $P(Y = y) > 0$,

$E(XY \mid Y = y) = yx_1P\{XY = yx_1 \mid Y = y\} + yx_2P\{XY = yx_2 \mid Y = y\} + ... + yx_nP\{XY = yx_n \mid Y = y\}$

$= y[x_1P\{X = x_1 \mid Y = y\} + x_2P\{X = x_2 \mid Y = y\} + ... + x_nP\{X = x_n \mid Y = y\}] = yE\{X \mid Y\}$

23.

Since X = Y a.e., and Y = Z a.e. then $P(X \ne Y) = 0$ and $P(Y \ne Z) = 0$.

We will show that $(X \ne Z)$ is a subset of $(X \ne Y) \cup (Y \ne Z)$.

To show this we will show $(X \neq Z) \subseteq [(X \neq Y)\cup(Y \neq Z)]'$

Step 1: $[(X \neq Y)\cup(Y \neq Z)]' = (X \neq Y)'\cap(Y \neq Z)' = (X = Y)\cap(Y = Z)$

Step 2: $(X \neq Z)\cap[(X \neq Y)\cup(Y \neq Z)]' = (X \neq Z)\cap[(X = Y)\cap(Y = Z)]$

Step 3: $[(X = Y)\cap(Y = Z)] \subseteq (X = Z)$

Step 4: Since $(X \neq Z) \cap(X = Z) = \phi$ and $[(X = Y)\cap(Y = Z)] \subseteq (X = Z)$, we have

$(X \neq Z) \cap[(X = Y)\cap(Y = Z)] = \phi = (X \neq Z) \cap[(X \neq Y)\cup(Y \neq Z)]'$

Step 5: Therefore, $(X \neq Z) \subseteq [(X \neq Y)\cup(Y \neq Z)]$

Step 6: From Step 5, $P(X \neq Z) \leq P(X \neq Y) + P(Y \neq Z) = 0 + 0 = 0$

Therefore, $P(X \neq Z) = 0$.

24.

$E(X|Z = z) = x_1P\{X = x_1| Z = z\} + x_2P\{X =x_1 | Z = z\} + ... + x_nP\{X = x_n | Z = z\}$

$E(Y|Z = z) = y_1P\{Y = y_1| Z = z\} + y_2P\{Y =y_1 | Z = z\} + ... + y_nP\{Y = y_n | Z = z\}$

Step 1: Define $\mathbf{A} = (X = Y)$. On $\mathbf{A}$, X and Y are identical . Since $\mathbf{A}' = (X \neq Y)$, $P(\mathbf{A}') = 0$ and therefore $P(\mathbf{A}) = 1$.

Step 2: We write $E(X|Z = z)$ as 2 partitions: the first partition will be the sum of all x = yvalues from $\mathbf{A}$; the second partition will be the sum of a $x \neq y$ from $\mathbf{A}'$. Therefore,

$E(X |Z = z) =$

$[x_1 P(X = x_1|Z = z) +...+x_kP(X = x_k| Z = z)] + [x_{k+1} P(X = x_{k+1} |Z = z) +...+x_nP(X = x_n| Z = z)]$

Since $P(\mathbf{A}') = 0$, $P(X = x_{k+1}| Z = z) = ...=P(X = x_n | Z = z) = 0$

$E(Y |Z = z) =$

$[y_1 P(X = y_1|Z = z) +...+y_kP(X = y_k| Z = z)] + [y_{k+1} P(Y = y_{k+1} |Z = z) +...+y_nP(Y = y_n| Z = z)]$

Since $P(\mathbf{A}') = 0$, $P(Y = y_{k+1}| Z = z) = ... = P(Y = y_n | Z = z) = 0$

Step 3: Therefore, $E(X |Z = z) = [x_1 P(X = x_1|Z = z) +...+x_kP(X = x_k| Z = z)] =$

$y_1 P(Y = y_1|Z = z) +...+y_kP(X = y_k| Z = z) = E(Y |Z = z)$

25.

$E(XY|Z = z) = x_1y_1P[(X = x_1)\cap(Y = y_1)|Z = z] + x_1y_2P[(X = x_1)\cap(Y = y_2)|Z = z] + ... +$

$x_1y_nP[(X = x_1)\cap(Y = y_n)|Z = z]) + x_2y_1P[(X = x_2)\cap(Y = y_1)|Z = z]$

$x_2y_2P[(X = x_2)\cap(Y = y_2)|Z = z] + ...+ x_2y_nP[(X = x_2)\cap(Y = y_n)|Z = z]) + ... +$

$x_my_1P[(X = x_m)\cap(Y = y_1)|Z = z]) + x_my_2P[(X = x_m)\cap(Y = y_2)|Z = z] +$

$x_my_3P[(X = x_m)\cap(Y = y_3)|Z = z] + ...+ x_my_nP[(X = x_m)\cap(Y = y_n)|Z = z]) =$

$x_1y_1P(X = x_1 |Z = z)P(Y = y_1 |Z = z)+ x_1y_2P(X = x_1 |Z = z)P(Y = y_2 |Z = z) + ... +$

$x_1y_nP(X = x_1 |Z = z)P(Y = y_n |Z = z)\sim+\sim x_2y_1P(X = x_2 |Z = z)P(Y = y_1 |Z = z) +$

$x_2y_2P(X = x_2 |Z = z)P(Y = y_2 |Z = z) + ...+ x_2y_nP(X = x_2 |Z = z)P(Y = y_n |Z = z) + ... +$

$x_my_1P(X = x_m |Z = z)P(Y = y_1 |Z = z) + x_my_2P(X = x_m |Z = z)P(Y = y_2 |Z = z) +$

$x_my_3P(X = x_m |Z = z)P(Y = y_3 |Z = z) + ...+ x_my_nP(X = x_m |Z = z)P(Y = y_n |Z = z) =$

$x_1P(X = x_1 |Z = z)\{y_1P(Y = y_1 |Z = z) + y_2P(Y = y_2 |Z = z) + y_nP(Y = y_n |Z = z)\} +$

$x_2P(X = x_2 |Z = z)\{y_1P(Y = y_1 |Z = z) + y_2P(Y = y_2 |Z = z) + ...+ y_nP(Y = y_n |Z = z)\} +$

$x_mP(X = x_m |Z = z)\{y_1P(Y = y_1 |Z = z) + y_2P(Y = y_2 |Z = z) + ...+ y_nP(Y = y_n |Z = z)\} =$

$x_1P(X = x_1 |Z = z)E(Y |Z = z) + x_2P(X = x_2)E(Y |Z = z) + ... + x_mP(X = x_m)E(Y |Z = z) =$

$[x_1P(X = x_1)+ x_2P(X = x_2) + x_mP(X = x_m)]E(Y) = E(X |Z = z)E(Y |Z = z).$