PROBABILITY THEORY
Lesson 14
Random Variables

14.1 - What is a Random Variable?

14.1 - Problem 1:

Sample Space	Random Variable X
(w,w)	0
(r,w)	1
(w,r)	1
(r,r)	2

Therefore, X(w,w) = 0, X(r,w) = 1, X(w,r) = 1, X(r,r) = 2.

14.1 - Problem 2:

Sample Space	Random Variable X
(Billy,Jane)	10
(Billy,Frank)	15
(Jane,Frank)	15

Therefore, X(Billy,Jane) = 10, X(Billy,Frank) = 15, X(Jane,Frank) = 15 .
Therefore,

X(2,2) = 4, X(2,4) = 6, X(2,6) = 8,

X(4,2) = 6, X(4,4) = 8, X(4,6) = 10,

X(6,2) = 8, X(6,4) = 10, X(6,6) = 12.

14.1 - Problem 3:

The following is a list of all possible groups of (10,15,10,20,30) containing 4 numbers:

(10,15,10,20), (10,15,10,30), (10,15,20,30), (10,10,20,30), (15,10,20,30).

Sample Space	**Random Variable X**
(10,15,10,20) (10,15,10,30) (10,15,20,30) (10,10,20,30) (15,10,20,30)	(10 + 15 + 10 + 20)/4 = 13.75 (10 + 15 + 10 + 30)/4 = 16.25 (10 + 15 + 20 + 30)/4 = 18.75 (10 + 10 + 20 + 30)/4 = 17.50 (15 + 10 + 20 + 30)/4 = 18.75

Sample Space	**Random Variable X**
(10,15,10,20)	13.75
(10,15,10,30)	16.25
(10,15,20,30)	18.75
(10,10,20,30)	17.50
(15,10,20,30)	18.75

Therefore,

X(10,15,10,20) = 13.75, X(10,15,10,30) = 16.25,

X(10,15,20,30) = 18.75, X(10,10,20,30) = 17.50, X(15,10,20,30) = 18.75

14.2 -What is a Probability Distribution of a Random Variable?

14.2 - Problem 1:

Step 1: The sample space is

S = {(g,g),(g,b),(b,g),(g,r),(r,g),(b,b),(b,r),(r,b),(r,r)}

#**S** = 9

Step 2: {X = $30} = {(b,b),(g,g),(r,r)}

P{X = $30} = P{(b,b),(g,g),(r,r)} = 3/9

{X = -$20} = {(b,b),(g,g),(r,r)}′

P{X = -$20} = P{(b,b),(g,g),(r,r)}′ = 1 - 3/9 = 6/9

Therefore,

x	**P{X = x}**
\$30	3/9
-\$20	6/9
	Total = 1

14.2 - Problem 2:

Step 1: The sample space is

S = {(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),
(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),
(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),
(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),
(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),
(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)}

#**S** = 36

{X = 0} = {(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)}

P{X = 0} = P{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)} = 6/36

{X = 1} = {(1,2),(2,1)}

P{X = 1} = P{(1,2),(2,1),(2,3),(3,2),(4,3),(3,4),(5,4),(4,5),(6,5),(5,6)} = 10/36

{X = 2} = {(1,3),(3,1),(2,4),(4,2),(3,5),(5,3),(6,4),(4,6)} = 8/36

P{X = 3} = P{(1,4),(4,1),(2,5),(5,2),(3,6),(6,3)} = 6/36

{X = 4} = {(1,5),(5,1),(6,2),(2,6)}

P{X = 4} = P{(1,5),(5,1),(6,2),(2,6)} = 4/36

{X = 5} = {(1,6),(6,1)}

P{X = 5} = P{(1,6),(6,1)} = 2/36

Therefore,

x	P{X = x}
0	6/36
1	10/36
2	8/36
3	6/36
4	4/36
5	2/36
	Total = 1

14.2 - Problem 3:

Step 1:
$\mathbf{W_1}$**:** The event he wins the first game.
$\mathbf{W_2}$: The event he wins the second game.
$\mathbf{W_3}$: The event he wins the third game.
$\mathbf{W_4}$: The event he wins the fourth game.

We assume the games are independent of each other.

Step 2: The event he wins all 4 games: $\mathbf{W_1 \cap W_2 \cap W_3 \cap W_4}$.

$\{X = \$400\} = \{\mathbf{W_1 \cap W_2 \cap W_3 \cap W_4}\}$

$P\{X = \$400\} = P(\mathbf{W_1 \cap W_2 \cap W_3 \cap W_4}) = 0.60^4 = 0.1296$

The event he wins three games:

$(W_1' \cap \mathbf{W_2} \cap \mathbf{W_3} \cap W_4) \cup (\mathbf{W_1} \cap \mathbf{W_2}' \cap \mathbf{W_3} \cap \mathbf{W_4}) \cup (\mathbf{W_1} \cap \mathbf{W_2} \cap \mathbf{W_3}' \cap W_4) \cup (\mathbf{W_1} \cap \mathbf{W_2} \cap \mathbf{W_3} \cap W_4')$

$P\{X = \$100 + \$100 + \$100 - \$110 = \$190\} =$

$P[(\mathbf{W_1}' \cap \mathbf{W_2} \cap \mathbf{W_3} \cap \mathbf{W_4}) \cup (\mathbf{W_1} \cap \mathbf{W_2}' \cap \mathbf{W_3} \cap \mathbf{W_4}) \cup (\mathbf{W_1} \cap \mathbf{W_2} \cap \mathbf{W_3}' \cap \mathbf{W_4}) \cup (\mathbf{W_1} \cap \mathbf{W_2} \cap \mathbf{W_3} \cap \mathbf{W_4}')] =$

$P(\mathbf{W_1}' \cap \mathbf{W_2} \cap \mathbf{W_3} \cap \mathbf{W_4}) + P(\mathbf{W_1} \cap \mathbf{W_2}' \cap \mathbf{W_3} \cap \mathbf{W_4}) + P(\mathbf{W_1} \cap \mathbf{W_2} \cap \mathbf{W_3}' \cap \mathbf{W_4}) + P(\mathbf{W_1} \cap \mathbf{W_2} \cap \mathbf{W_3} \cap \mathbf{W_4}') =$

$(0.4)(0.6)(0.6)(0.6) + (0.6)(0.4)(0.6)(0.6) + (0.6)(0.6)(0.4)(0.6) + (0.6)(0.6)(0.6)(0.4) =$

$4(0.4)(0.6)^3 = 0.3456$.

The event he wins two games:

$(\mathbf{W_1}' \cap \mathbf{W_2}' \cap \mathbf{W_3} \cap \mathbf{W_4}) \cup (\mathbf{W_1}' \cap \mathbf{W_2} \cap \mathbf{W_3}' \cap W_4) \cup (\mathbf{W_1}' \cap \mathbf{W_2} \cap \mathbf{W_3} \cap W_4') \cup (\mathbf{W_1} \cap \mathbf{W_2}' \cap \mathbf{W_3}' \cap W_4) \cup$

$(W_1 \cap W_2' \cap W_3 \cap W_4') \cup (W_1 \cap W_2 \cap W_3' \cap W_4')$

$P\{X = \$100 + \$100 - \$110 - \$110 = -\$20\} =$

$P[(W_1' \cap W_2' \cap W_3 \cap W_4) \cup (W_1' \cap W_2 \cap W_3' \cap W_4) \cup (W_1' \cap W_2 \cap W_3 \cap W_4') \cup (W_1 \cap W_2' \cap W_3' \cap W_4) \cup$

$(W_1 \cap W_2' \cap W_3 \cap W_4') \cup (W_1 \cap W_2 \cap W_3' \cap W_4')] =$

$P[(W_1' \cap W_2' \cap W_3 \cap W_4) + P(W_1' \cap W_2 \cap W_3' \cap W_4) + P(W_1' \cap W_2 \cap W_3 \cap W_4') + P(W_1 \cap W_2' \cap W_3' \cap W_4)$

$+ P(W_1 \cap W_2' \cap W_3 \cap W_4')\ P(W_1 \cap W_2 \cap W_3' \cap W_4')] =$

$= (0.4)(0.4)(0.6)(0.6) + (0.4)(0.6)(0.4)(0.6) + (0.4)(0.6)(0.6)(0.4) +$

$(0.6)(0.4)(0.4)(0.6) + (0.6)(0.4)(0.6)(0.4) + (0.6)(0.6)(0.4)(0.4) = 6(0.6)^2(0.4)^2 = 0.3456$

The event he wins one games:

$(\mathbf{W}_1' \cap \mathbf{W}_2' \cap \mathbf{W}_3' \cap \mathbf{W}_4) \cup (\mathbf{W}_1' \cap \mathbf{W}_2' \cap \mathbf{W}_3 \cap \mathbf{W}_4') \cup (\mathbf{W}_1' \cap \mathbf{W}_2 \cap \mathbf{W}_3' \cap \mathbf{W}_4') \cup (\mathbf{W}_1 \cap \mathbf{W}_2' \cap \mathbf{W}_3' \cap \mathbf{W}_4')$

$P\{X = \$100 - \$110 - \$110 - \$110 = -\$230\} =$

$P\{(\mathbf{W}_1' \cap \mathbf{W}_2' \cap \mathbf{W}_3' \cap \mathbf{W}_4) \cup (\mathbf{W}_1' \cap \mathbf{W}_2' \cap \mathbf{W}_3 \cap \mathbf{W}_4') \cup (\mathbf{W}_1' \cap \mathbf{W}_2 \cap \mathbf{W}_3' \cap \mathbf{W}_4') \cup (\mathbf{W}_1 \cap \mathbf{W}_2' \cap \mathbf{W}_3' \cap \mathbf{W}_4')\}$
$=$
$P(\mathbf{W}_1' \cap \mathbf{W}_2' \cap \mathbf{W}_3' \cap \mathbf{W}_4) + P(\mathbf{W}_1' \cap \mathbf{W}_2' \cap \mathbf{W}_3 \cap \mathbf{W}_4') + P(\mathbf{W}_1' \cap \mathbf{W}_2 \cap \mathbf{W}_3' \cap \mathbf{W}_4') +$

$P(W_1' \cap W_2' \cap W_3' \cap W_4')\} =$

$(0.4)(0.4)(0.4)(0.6) + (0.4)(0.4)(0.6)(0.4) + (0.4)(0.6)(0.4)(0.4) +$

$(0.6)(0.4)(0.4)(0.4) = 4(0.4)^3(0.6) = 0.1536.$

The event he wins no games: $(W_1' \cap W_2' \cap W_3' \cap W_4')$

$P\{X = -\$440\} = P(W_1' \cap W_2' \cap W_3' \cap W_4') = 0.4^4 = 0.0256$

Therefore,

x	**P{X = x}**
\$400	0.1296
\$190	0.3456
-\$20	0.3456
-\$230	0.1536
-\$440	0.0256
	Total 1

14.3 - Another Way to Define the Probability Distribution of A Random Variable.

14.3 - Problem 1:

Step 1:
$\mathbf{C_1}$: the event a club was drawn on the first drawing.
$\mathbf{C_2}$: the event a club was drawn on the second drawing.

Step 1: No club was drawn:

$\mathbf{C_1}'\cap\mathbf{C_2}' = \{X = 0\} = \{X \le 0\}$

$P\{X \le 0\} = P(\mathbf{C_1}'\cap\mathbf{C_2}') = P(\mathbf{C_1}')P(\mathbf{C_2}'|\mathbf{C_1}') = (\frac{39}{52})(\frac{38}{51}) = \frac{1482}{2652}$

Step 2: One club was drawn:

$(\mathbf{C_1}'\cap\mathbf{C_2})\cup(\mathbf{C_1}\cap\mathbf{C_2}')$

$P\{X = 1\} = P\{(\mathbf{C_1}'\cap\mathbf{C_2})\cup(\mathbf{C_1}\cap\mathbf{C_2}')\} = P(\mathbf{C_1}'\cap\mathbf{C_2}) + P(\mathbf{C_1}\cap\mathbf{C_2}') =$

$P(\mathbf{C_1}')P(\mathbf{C_2}|\mathbf{C_1}') + P(\mathbf{C_1})P(\mathbf{C_2}'|\mathbf{C_1}) = (\frac{39}{52})(\frac{13}{51}) + (\frac{13}{52})(\frac{39}{51}) = \frac{1014}{2652}$

$P\{X \le 1\} = P\{X = 0\} + P\{X = 1\} = \frac{1482}{2652} + \frac{1014}{2652} = \frac{2496}{2652}$

Since the maximum number of clubs possible: $P\{X \le 2\} = 1$

Therefore,

x	**$P\{X \le x\}$**
0	$\frac{1482}{2652}$
1	$\frac{2496}{2652}$
2	1

14.3 Problem 2:

Step 1:
$\mathbf{R}_1$: The event the red side occurs on the first toss.
$\mathbf{R}_2$: The event the red side occurs on the second toss.

$P(\mathbf{R}_1) = P(\mathbf{R}_2) = 1/6$.

Step 2: The event the red side never appears: $\mathbf{R}_1' \cap \mathbf{R}_2'$.

$P\{X \le 0\} = P\{X = 0\} = P\{\mathbf{R}_1' \cap \mathbf{R}_2'\} = (5/6)(5/6) = 25/36$

The event the red side occurs only once: $(\mathbf{R}_1 \cap \mathbf{R}_2') \cup (\mathbf{R}_1' \cap \mathbf{R}_2)$

$$P\{X = 1\} = P[(\mathbf{R}_1 \cap \mathbf{R}_2') \cup (\mathbf{R}_1' \cap \mathbf{R}_2)] = P(\mathbf{R}_1 \cap \mathbf{R}_2') + P(\mathbf{R}_1' \cap \mathbf{R}_2) = 2\left(\frac{1}{6}\right)\left(\frac{5}{6}\right) = \frac{10}{36}$$

$$P\{X \le 1\} = P\{X = 0\} + P\{X = 1\} = \frac{25}{36} + \frac{10}{36} = \frac{35}{36}$$

The maximum number of times the red side can occur is 2. Therefore, $P\{X \le 2\} = 1$

Therefore,

x	P{X ≤ x}
0	25/36
1	35/36
2	1

14.3 - Problem 3:

The winnings for each possible outcome are \$200, -\$10, -\$220. Except for the values of x, this problem is the same as Example 3.3. Therefore, we shall use the following results previously derived:

$P\{X = \$200\} = 0.36$

$P\{X = -\$10\} = 0.48$

$P\{X = -\$220\} = 0.16$

$x = -\$300$:

$P\{x \le -\$300\} = 0$, since the smallest loss possible is -\$220.

x = -$200:

P{x ≤ -$200} = P{x = -$200} + P{X = -$220} = 0 + 0.16 = 0.16.

X = -$100:

P{x ≤ -$100} = P{x = -$100} + P{X = -$220} = 0 + 0.16 = 0.16

x = $0:

P{x ≤ $0} = P{x = $0} + P{x = -$10} + P{x = -$220} = 0 + 0.48 + 0.16 = 0.64

x = $100:

P{x ≤ $100} = P{x = $100} + P{x = -$10} + P{x = -$220} = 0 + 0.48 + 0.16 = 0.64

x = $200:

P{x ≤ $200} = P{x = $200} + P{x = -$10} + P{x = -$220} = 0.48 + 0.16 + 0.36 = 1

x = 300:

P{x ≤ $300} = P{x = 300} + P{x = $200} + P{x = -$10} + P{x = -$220} =

0 + 0.48 + 0.16 + 0.36 = 1

Therefore, we have the table:

x	**P{X ≤ x}**
-$300	0.0
-$200	0.16
-$100	0.16
$0	0.64
$100	0.64
$200	1.00
$300	1.00

14.3 - Problem 4:

P{X = 0}= P{X ≤ 0} = 0.15

$P\{X \le 1\} = P[\{X \le 0\} \cup \{X = 1\}] = P\{X \le 0\} + P\{X = 1\}$

$P\{X = 1\} = P\{X \le 1\} - P\{X \le 0\} = 0.15 - 0.15 = 0$

$P\{X \le 2\} = P[\{X \le 1\} \cup \{X = 2\}] = P\{X \le 1\} + P\{X = 2\}$

$P\{X = 2\} = P\{X \le 2\} - P\{X \le 1\} = 0.75 - 0.15 = 0.60$

$P\{X \le 3\} = P[\{X \le 2\} \cup \{X = 3\}] = P\{X \le 2\} + P\{X = 3\}$

$P\{X = 3\} = P\{X \le 3\} - P\{X \le 2\} = 1 - 0.75 = 0.25$

x	**P{X = x}**
0	0.15
1	0.0
2	0.60
3	0.25
	Total 1

Supplementary Problems

1.
Step 1: $\{X \le k + 1\} = \{X = k + 1\} \cup \{X \le k\}$

Step 2: $P\{X \le k + 1\} = P[\{X = k + 1\} \cup \{X \le k\}] = P\{X = k + 1\} + P\{X \le k\}$

$P\{X \le k + 1\} = P\{X = k + 1\} + P\{X \le k\}$

Step 3: $P\{X = k + 1\} = P\{X \le k + 1\} - P\{X \le k\}$

2.
➤ a.
Since X = x for x = 0,1,2,3,4,5 only, $P\{X \ge 0\} = 1$.

➤ b.
$P\{X \ge 1\} = 1 - P\{X < 1\} = 1 - P\{X \le 0\} = 1 - 0.2 = 0.8$

➤ c.
$P\{X \ge 2\} = 1 - P\{X < 2\} = 1 - P\{X \le 1\} = 1 - 0.3 = 0.7$

➤ d.
$P\{X = 3\} = P\{X \le 3\} - P\{X \le 2\} = 0.50 - 0.35 = 0.15$

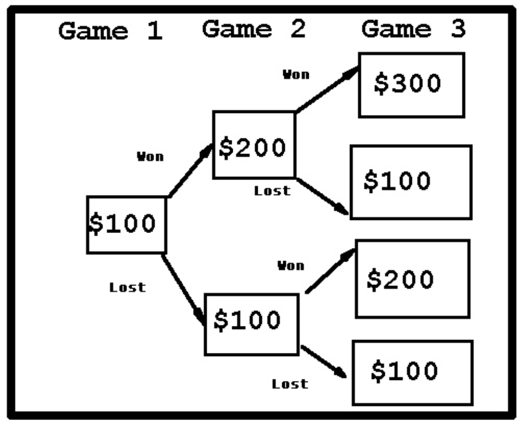
➤ e.
$P\{2 \le X \le 4\} = P\{X \le 4\} - P\{X \le 1\} = 0.70 - 0.30 = 0.40$

➤ f.
$P\{2 < X \le 4\} = P\{3 \le X \le 4\} =$

$P\{X \le 4\} - P\{X \le 2\} = 0.70 - 0.35 = 0.35$

➤ g.
$P\{X > 4\} = 1 - P\{X \le 4\} = 1 - 0.70 = 0.30$

3.
$\mathbf{W_1}$: The event that he wins the first game.
$\mathbf{W_2}$: The event that he wins the second game.
$\mathbf{W_3}$: The event that he wins the third game.



Case 1: He wins all three games: $\mathbf{W_1 \cap W_2 \cap W_3}$.

Step 1: Since he bet \$100 on the first game, he won \$90.

Step 2: Since he bet \$200 on the second game, he won an additional \$180.

Step 3: Since he bet \$300 on the third game, he won an additional \$270.

Step 4: The total is \$90 + \$180 + \$270 = \$540

Step 5: $P(\mathbf{W_1 \cap W_2 \cap W_3}) = (0.60)^3 = 0.216$

Step 6: $P\{X = \$540\} = 0.216$

Case 2: He wins the first two games but losses the third: $\mathbf{W_1 \cap W_2 \cap W_3'}$.

Step 1: Since he bet \$100 on the first game, he won \$90.

Step 2: Since he bet \$200 on the second game, he won an additional \$180.

Step 3: Since he bet \$300 on the third game, he lost \$300 on this game .

Step 4: The total is \$90 + \$180 - \$300 = -\$30.

Step 5: $P(\mathbf{W_1 \cap W_2 \cap W_3'}) = (0.6)^2(0.4) = 0.144$

Step 6: P(X = -\$30) = 0.144

Case 3: He wins the first and third games but losses the second: $\mathbf{W_1} \cap \mathbf{W_2}' \cap \mathbf{W_3}$.

Step 1: Since he bet \$100 on the first game, he won \$90.

Step 2: Since he bet \$200 on the second game and lost, he losses -\$200 on this game.

Step 3: Since he bet \$100 on the third game and won, he wins \$90 on this game .

Step 4: The total is \$90 + -\$200 + \$90 = -\$20.

Step 5: $P(\mathbf{W_1} \cap \mathbf{W_2}' \cap \mathbf{W_3}) = (0.6)^2(0.4) = 0.144$

Step 6: P(X = -\$20) = 0.144

Case 4: He losses the first game but wins the other two games: $\mathbf{W_1}' \cap \mathbf{W_2} \cap \mathbf{W_3}$.

Step 1: Since he bet \$100 on the first game and losses, he lost -\$100.

Step 2: Since he bet \$100 on the second game and won, he wins \$90 on this game.

Step 3: Since he bet \$200 on the third game and won, he wins \$180 on this game .

Step 4: The total is -\$100 + \$90 + \$180 = \$170.

Step 5: $P(\mathbf{W_1}' \cap \mathbf{W_2} \cap \mathbf{W_3}) = (0.40)(0.6)^2 = 0.144$

Step 6: P(X = \$170) = 0.144

Case 5: He losses the first two games but wins the third game: $\mathbf{W_1}' \cap \mathbf{W_2}' \cap \mathbf{W_3}$.

Step 1: Since he bet \$100 on the first game and losses, he lost -\$100.

Step 2: Since he bet \$100 on the second game and loses, he lost -\$100 on this game.

Step 3: Since he bet \$100 on the third game and won, he wins \$90 on this game .

Step 4: The total is -\$100 + -\$100 + \$90 = \$170.

Step 5: $P(\mathbf{W_1}' \cap \mathbf{W_2}' \cap \mathbf{W_3}) = (0.40)^2(0.6) = 0.096$

Step 6: P(X = -\$110) = 0.096

Case 6: He losses the first and third games but wins the second game: $\mathbf{W_1}'\cap\mathbf{W_2}\cap\mathbf{W_3}'$.

Step 1: Since he bet \$100 on the first game and losses, he lost -\$100.

Step 2: Since he bet \$100 on the second game and wins, he wins \$90 on this game.

Step 3: Since he bet \$200 on the third game and losses, he losses -\$200 on this game .

Step 4: The total is -\$100 + \$90 - \$200 = -\$210.

Step 5: $P(\mathbf{W_1}'\cap\mathbf{W_2}\cap\mathbf{W_3}') = (0.40)^2(0.6) = 0.096$

Case 7: He wins the first game but losses the other two games: $\mathbf{W_1}\cap\mathbf{W_2}'\cap\mathbf{W_3}'$.

Step 1: Since he bet \$100 on the first game and wins, he wins \$90.

Step 2: Since he bet \$200 on the second game and losses, he losses -\$200 on this game.

Step 3: Since he bet \$100 on the third game and losses, he losses -\$100 on this game .

Step 4: The total is \$90 + -\$200 - \$100 = -\$210.

Step 5: $P(\mathbf{W_1}\cap\mathbf{W_2}'\cap\mathbf{W_3}') = (0.60)(0.40)^2 = 0.096$

Step 6: $P(X = -\$210) = P\{(\mathbf{W_1}'\cap\mathbf{W_2}\cap\mathbf{W_3}')\cup(\mathbf{W_1}\cap\mathbf{W_2}'\cap\mathbf{W_3}')\} =$

$0.096 + 0.096 = 0.192$

Case 8: He losses all three games: $\mathbf{W_1}'\cap\mathbf{W_2}'\cap\mathbf{W_3}'$.

$P\{X = -\$300\} = P(\mathbf{W_1}'\cap\mathbf{W_2}'\cap\mathbf{W_3}') = (0.4)^3 = 0.064$

The distribution of X is therefore,

x	P{X = x}
\$540	0.216
- 30	0.144
- 20	0.144
170	0.144
-110	0.096
-210	0.192
-300	0.064
	Total = 1

4.

➤(a).

$\{X \geq 3\}$: The event that at least 3 cars are sold.

$P\{X \geq 3\} = 0.20 + 0.10 + 0.05 = 0.35$, the probability that he sell at least 3 cars.

➤(b).

$\{X \leq 3\}$: The event at most 3 cars.

$P\{X \leq 3\} = 0.20 + 0.30 + 0.25 + 0.10 = 0.85$, the probability that at most 3 cars are sold.

➤(c).
$\{ X = 3\} \cup \{ X = 5\}$: The event 3 or 5 cars were sold

$P[\{ X = 3\} \cup \{ X = 5\}] = P\{ X = 3\} + P\{ X = 5\} = 0.20 + 0.05 = 0.25$

➤(d).
$\{1 \leq X \leq 4\}$: The event that between 1 and 4 cars were sold.

$P\{1 \leq X \leq 4\} = 0.25 + 0.30 + 0.20 + 0.10 = 0.85$

➤(e).
$P\{X < 5\} = \{ X \leq 4\}$: The event less than 5 cars.

$P\{ X < 5\} = P\{ X \leq 4\} = 0.10 + 0.25 + 0.30 + 0.20 + 0.10 = 0.95$

5.
S =
{(1,1),(1,2),(2,1),{(1,3),(2,2),(3,1),(1,4),(2,3),(3,2),(4,1),(1,5),(2,4),(3,3),(4,2),(5,1),(1,6),(2,5),
(3,4),(4,3),(5,2),(6,1),(2,6),(3,5),(4,4),(5,3),(6,2),(3,6),(4,5),(5,4),(6,3),(4,6),(5,5),(6,4),(5,6),(6,5),
(6,6)

$X(1,1) = 2X_1 - X_2 = 2(1) - 1 = 1$ $X(1,2) = 2X_1 - X_2 = 2(1) - 2 = 0$
$X(1,3) = 2X_1 - X_2 = 2(1) - 3 = -1$ $X(1,4) = 2X_1 - X_2 = 2(1) - 4 = -2$
$X(1,5) = 2X_1 - X_2 = 2(1) - 5 = -3$ $X(1,6) = 2X_1 - X_2 = 2(1) - 6 = -4$

$X(2,1) = 2X_1 - X_2 = 2(2) - 1 = 3$ $X(2,2) = 2X_1 - X_2 = 2(2) - 2 = 2$
$X(2,3) = 2X_1 - X_2 = 2(2) - 3 = 1$ $X(2,4) = 2X_1 - X_2 = 2(2) - 4 = 0$
$X(2,5) = 2X_1 - X_2 = 2(2) - 5 = -1$ $X(2,6) = 2X_1 - X_2 = 2(2) - 6 = -2$

$X(3,1) = 2X_1 - X_2 = 2(3) - 1 = 5$ $X(3,2) = 2X_1 - X_2 = 2(3) - 2 = 4$
$X(3,3) = 2X_1 - X_2 = 2(3) - 3 = 3$ $X(3,4) = 2X_1 - X_2 = 2(3) - 4 = 2$
$X(3,5) = 2X_1 - X_2 = 2(3) - 5 = 1$ $X(3,6) = 2X_1 - X_2 = 2(3) - 6 = 0$

$X(4,1) = 2X_1 - X_2 = 2(4) - 1 = 7$ $X(4,2) = 2X_1 - X_2 = 2(4) - 2 = 6$
$X(4,3) = 2X_1 - X_2 = 2(4) - 3 = 5$ $X(4,4) = 2X_1 - X_2 = 2(4) - 4 = 4$
$X(4,5) = 2X_1 - X_2 = 2(4) - 5 = 3$ $X(4,6) = 2X_1 - X_2 = 2(4) - 6 = 2$

$X(5,1) = 2X_1 - X_2 = 2(5) - 1 = 9$ $X(5,2) = 2X_1 - X_2 = 2(5) - 2 = 8$

$X(5,3) = 2X_1 - X_2 = 2(5) - 3 = 7$ $X(5,4) = 2X_1 - X_2 = 2(5) - 4 = 6$

$X(5,5) = 2X_1 - X_2 = 2(5) - 5 = 5$ $X(5,6) = 2X_1 - X_2 = 2(5) - 6 = 4$

$X(6,1) = 2X_1 - X_2 = 2(6) - 1 = 11$ $X(6,2) = 2X_1 - X_2 = 2(6) - 2 = 10$

$X(6,3) = 2X_1 - X_2 = 2(6) - 3 = 9$ $X(6,4) = 2X_1 - X_2 = 2(6) - 4 = 8$

$X(6,5) = 2X_1 - X_2 = 2(6) - 5 = 7$ $X(6,6) = 2X_1 - X_2 = 2(6) - 6 = 6$

$P\{X = -4\} = P\{(1,6)\} = 1/36$

$P\{X = -3\} = P\{(1,5)\} = 1/36$

$P\{X = -2\} = P\{(1,4),(2,6)\} = 2/36$

$P\{X = -1\} = P\{(1,3),(2,5)\} = 2/36$

$P\{X = 0\} = P\{(1,2),(2,4),(3,6)\} = 3/36$

$P\{X = 1\} = P\{(1,1),(2,3),(3,5)\} = 3/36$

$P\{X = 2\} = P\{(2,2),(3,4),(4,6)\} = 3/36$

$P\{X = 3\} = P\{(2,1),(3,3),(4,5)\} = 3/36$

$P\{X = 4\} = P\{(3,2),(4,4),(5,6)\} = 3/36$

$P\{X = 5\} = P\{(3,1),(4,3),(5,5)\} = 3/36$

$P\{X = 6\} = P\{(4,2),(5,4),(6,6)\} = 3/36$

$P\{X = 7\} = P\{(4,1),(5,3),(6,5)\} = 3/36$

$P\{X = 8\} = P\{(5,2),(6,4)\} = 2/36$

$P\{X = 9\} = P\{(5,1),(6,3)\} = 2/36$

$P\{X = 10\} = P\{(6,2)\} = 1/36$

$P\{X = 11\} = P\{(6,1)\} = 1/36$

The distribution table is

x	**P{X =2X$_1$ - X$_2$}**
-4	1/36
- 3	1/36
- 2	2/36
-1	2/36
0	3/38

1	3/36
2	3/36
3	3/36
4	3/36
5	3/36
6	3/36
7	3/36
8	2/36
9	2/36
10	1/36
11	1/36
	Total = 1

6.
From 14.2 - Example 2, we have the distribution

x	P{X = x}
2	1/36
3	2/36
4	3/36
5	4/36
6	5/36
7	6/36
8	5/36
9	4/36
10	3/36
11	2/36
12	1/36
	Total 1

$P\{X \geq 2\} = 1$, since the smallest value for the sum is 2.

$P\{X \geq 3\} = 1 - P\{X \leq 2\} = 1 - P\{X = 2\} = 1 - 1/36 = 35/36$

$P\{X \geq 4\} = 1 - P\{X \leq 3\} = 1 - 2/36 - 1/36 = 33/36$

$P\{X \geq 5\} = 1 - P\{X \leq 4\} = 1 - 3/36 - 2/36 - 1/36 = 30/36$

$P\{X \geq 6\} = 1 - P\{X \leq 5\} = 1 - 4/36 - 3/36 - 2/36 - 1/36 = 26/36$

$P\{X \geq 7\} = 1 - P\{X \leq 6\} = 1 - 5/36 - 4/36 - 3/36 - 2/36 - 1/36 = 21/36$

$P\{X \geq 8\} = 1 - P\{X \leq 7\} = 1 - 6/36 - 5/36 - 4/36 - 3/36 - 2/36 - 1/36 = 15/36$

$P\{X \geq 9\} = 1 - P\{X \leq 8\} = 1 - 5/36 - 6/36 - 5/36 - 4/36 - 3/36 - 2/36 - 1/36 = 10/36$

$P\{X \geq 10\} = 1 - P\{X \leq 9\} = 1 - 4/36 - 5/36 - 6/36 - 5/36 - 4/36 - 3/36 - 2/36 - 1/36 = 6/36$

$P\{X \geq 11\} = 1 - P\{X \leq 10\} = 1 - 3/36 - 4/36 - 5/36 - 6/36 - 5/36 - 4/36 - 3/36 - 2/36 - 1/36 = 3/36$

$P\{X \geq 12\} = 1 - P\{X \leq 11\} = 1 - 2/36 - 3/36 - 4/36 - 5/36 - 6/36 - 5/36 - 4/36 - 3/36 - 2/36 - 1/36$
$= 1/36$

The distribution table is

X = k	**P(X ≥ k)**
2	36/36
3	35/36
4	33/36
5	30/46
6	26/36
7	21/36
8	15/36
9	10/36
10	6/36
11	3/36
12	1/36

7.
S = {(h,h), (h,t,h), (t,h,h), (h,t,t,h), (t,h,t,h), (t,t,h,h), (t,t,t,t,t), (h,t,t,t,t), (t,h,t,t,t), (t,t,h,t,t), (t,t,t,h,t), (t,t,t,t,h), (h,t,t,t,h), (t,h,t,t,h), (t,t,h,t,h), (t,t,t,h,h)}

#**S** = 16
P{X = 2} = P{(h,h)} = 1/16

P{X = 3} = P{ (h,t,h), (t,h,h),} = 2/16

P{X = 4} = P{(h,t,t,h), (t,h,t,h), (t,t,h,h)} = 3/16

P{X = 5} = P{(t,t,t,t,t), (h,t,t,t,t), (t,h,t,t,t), (t,t,h,t,t), (t,t,t,h,t), (t,t,t,t,h), (h,t,t,t,h), (t,h,t,t,h), (t,t,h,t,h), (t,t,t,h,h)}} = 10/16

x	**P{X = x}**
2	1/16
3	2/16
4	3/16
5	10/16

8.

$$\frac{P[(X = x)\cap(Y = y)]}{P(Y = y)} = \frac{\#[(X = x)\cap(Y = y)]}{\#[(Y = y)]}$$

For Y = 0, we assume no queen is selected. Therefore, #(Y = 0) = (48)47 = 2256.

$$P(X = 0|Y = 0) = (\frac{44}{48})(\frac{43}{47}) = \frac{1892}{2256}$$

$$P(X = 1|Y = 0) = P(\mathbf{K}_1\cap\mathbf{K}_2') + P(\mathbf{K}_1'\cap\mathbf{K}_2) = (\frac{4}{48})(\frac{44}{47}) + (\frac{44}{48})(\frac{4}{47}) = \frac{352}{2256}$$

$$P(X = 2|Y = 0) = P(\mathbf{K}_1\cap\mathbf{K}_2) = (\frac{4}{48})(\frac{3}{47}) = \frac{12}{2256}$$

For Y = 1, we assume 1 queen is selected. Therefore, #(Y = 1) = (4)(48) + (48)(4) = 384.

$$P(X = 0|Y = 1) = \frac{\#[(\mathbf{K_1}'\cap\mathbf{Q_1}'\cap\mathbf{Q_2})\cup(\mathbf{Q_1}\cap\mathbf{K_2}'\cap\mathbf{Q_2}')]}{384} = \frac{(44)(4) + (4)(44)}{384} = \frac{352}{384}.$$

$$P(X = 1|Y = 1) = \frac{\#[(\mathbf{K_1}\cap\mathbf{Q_2})\cup(\mathbf{K_2}\cap\mathbf{Q_1})]}{384} = \frac{(4)(4) + (4)(4)}{384} = \frac{32}{384}.$$

$P(X = 2|Y = 1) = 0.$

For Y = 2, we assume 2 queen are selected.

$P(X = 0|Y = 2) = 1.$
$P(X = 1|Y = 2) = 0$
$P(X = 2|Y = 2) = 0.$

9.

➤a.

Since

$\{X = -2\} = \{X^3 = (-2)^3 = -8\}$, $\{X = -1\} = \{X = -1\} = \{X^3 = (-1)^3 = -1\}$,

$\{X = 1\} = \{X^3 = (1)^3 = 1\}$,$\{X = 2\} = \{X^3 = (2)^3 = 8\}$,

we have $P\{X = -2\} = P\{X^3 =-8\}$,$P\{X = -1\} = P\{X^3 =-1\}$,$P\{X = 1\} = P\{X^3 = 1\}$,$P\{X = 2\} =$

$P\{X^3 =8\}$.

Therefore, we have

$X^3 = x$	$P\{X = x\}$
-8	1/10
-1	2/10
1	3/10
8	4/10
	Total 1

➤b.
We have

$\{X^4 = (-2)^4 = 16\} = \{X = -2\} \cup \{X = 2\}$

$P\{X^4 = (-2)^4 = 16\} = P[\{X = -2\} \cup \{X = 2\}] = P\{X = -2\} + P\{X = 2\} = 1/10 + 4/10 = 5/10 = 1/2$

$\{X^4 = (-1)^4 = 1\} = \{X = -1\} \cup \{X = 1\}$

$P\{X^4 = (-1)^4 = 1\} = P[\{X = -1\} \cup \{X = 1\}] = P\{X = -1\} + P\{X = 1\} = 2/10 + 3/10 = 5/10 = 1/2$

$X^4 = x$	$P\{X = x\}$
1	1/2
16	1/2
	Total 1

➤c.
$\{X^4 - X^3 = (-2)^4 - (-2)^3 = 24\} = \{X = -2\}$

$P\{X^4 - X^3 = (-2)^4 - (-2)^3 = 24\} = P\{X = -2\} = 1/10$

$\{X^4 - X^3 = (-1)^4 - (-1)^3 = 2\} = \{X = 2\}$

$P\{X^4 - X^3 = (-1)^4 - (-1)^3 = 2\} = P\{X = -1\} = 2/10$

$\{X^4 - X^3 = (1)^4 - (1)^3 = 0\} = \{X = 1\}$

$P\{X^4 - X^3 = (1)^4 - (1)^3 = 0\} = P\{X = 1\} = 3/10$

$\{X^4 - X^3 = (2)^4 - (2)^3 = 8\} = \{X = 2\}$

$P\{X^4 - X^3 = (2)^4 - (2)^3 = 8\} = P\{X = 2\} = 3/10$

$X^4 - X^3 = x$	$P\{X = x\}$
24	1/10
2	2/10
0	3/10
8	4/10
	Total 1

10.

.
$P\{X = 0; Y = 0\} = P\{(\mathbf{K}_1'\cap\mathbf{Q}_1')\cap(\mathbf{K}_2'\cap\mathbf{Q}_2')\} = (44/52)(43/51) = 1892/2652$

$P\{X = 0; Y = 1\} = P\{(\mathbf{K}_1'\cap\mathbf{Q}_1')\cap\mathbf{Q}_2\} + P\{\mathbf{Q}_1 \cap(\mathbf{K}_2'\cap\mathbf{Q}_2')\} = (44/52)(4/51) + (4/52)(44/52) = 352/2652$

$P\{X = 1; Y = 0\} = P\{(\mathbf{K}_2'\cap\mathbf{Q}_2')\cap\mathbf{K}_1\} + P\{\mathbf{K}_2 \cap(\mathbf{K}_1'\cap\mathbf{Q}_1')\} = (4/52)(44/51) + (4/52)(44/52) = 352/2652$

$P\{X = 1; Y = 1\} = P\{(\mathbf{K}_1\cap\mathbf{Q}_2\} + P\{\mathbf{Q}_1 \cap\mathbf{K}_2\} = (4/52)(4/51) + (4/52)(4/52) = 32/2652$

$P\{X = 2; Y = 0\} = P\{\mathbf{K}_1\cap\mathbf{K}_2\} = (4/52)(3/51) = 12/2652$

$P\{X = 0; Y = 2\} = P\{(\mathbf{Q}_1\cap\mathbf{Q}_2\} = (4/52)(3/51) = 12/2652$

		Y 0	1	2	**X**
X	0	1892/2652	352/2652	12/2652	2256/2652
	1	352/2652	32/2652	0	384/2652
Y		2256/2652	384/2652	12/2652	**Total 1**

11.

		Y **0**	**1**	**2**	**3**	**X**
X	**1**	2p	0	0	p	3p
	2	6p	6p	6p	0	18p
	3	0	6p	0	0	6p
Y		8p	12p	6p	p	

$3p + 18p + 6p = 27p = 1$

$p = 1/27$

$8p + 12p + 6p + p = 27p = 1$

$p = 1/27$

		Y 0	1	2	3	X
X	**1**	2p	0	0	p	3/27
	2	6p	6p	6p	0	18/27
	3	0	6p	0	0	6/27
Y		8/27	12/27	6/27	1/27	

12.

➤a.

Step 1: **F**: The event that a 5 is rolled.

Step 2: $\{X = 1\} = \mathbf{F}_1$

$\{X = k\} = \mathbf{F}_1{}'\cap\mathbf{F}_2{}'\cap \ldots\cap\mathbf{F}_{k-1}{}'\cap\mathbf{F}_k$

Step 3: $P\{X = k\} = P(\mathbf{F}_1{}'\cap\mathbf{F}_2{}'\cap \ldots\cap\mathbf{F}_{k-1}{}'\cap\mathbf{F}_k) = P(\mathbf{F}_1{}')P(\mathbf{F}_2{}')\ldots P(\mathbf{F}_{k-1}{}')P(\mathbf{F}_k)$

$= (5/6)(5/6)\ldots(5/6)(1/6) = (5/6)^{k-1}(1/6)$

➤b.

Step 1: $\{X \le N\} = \{X = 1\}\cup\{X = 2\}\cup\ldots\cup\{X = N\}$

Step 2: $P\{X \le N\} = P\{X = 1\} + P\{X = 2\} + \ldots + P\{X = N\} =$

$1/6 + (5/6)(1/6) + (5/6)^2(1/6) + \ldots + (5/6)^{N-1}(1/6) = 1/6[1 + 5/6 + (5/6)^2 + \ldots + (5/6)^{N-1}]$

Step 3: $P\{X \le N\} = (1/6)[1 + 5/6 + (5/6)^2 + \ldots + (5/6)^{N-1}] =$

$(1/6)\{[1 + 5/6 + (5/6)^2 + \ldots + (5/6)^{N-1}][1- 5/6]\}/[1 - 5/6] = (1/6)[1 - (5/6)^N]/[1/6] = 1 - (5/6)^N$

Step 4: $P\{X \le N\} = 1 - (5/6)^N \ge 0.90$

$- (5/6)^N \ge -0.10$

$(5/6)^N \le 0.10$

$\text{Log}_{10}(5/6)^N \le \log_{10}(0.10)$

$N\log_{10}(5/6) \le \log_{10}(0.10)$

Using a calculator, $\log_{10}(5/6) \approx -0.08$ and $\log_{10}(0.10) = -1$

Step 5: $-0.08N \le -1$

$N \ge 1/0.08$

$N \ge 12.5$

Therefore the minimum number of rolls is 13.

➤c.
Step 1: $\{X \le N\}' = \{X > N\}$

Step 2: $P\{X > N\} = P\{X \le N\}' = 1 - P\{X \le N\}$

Step 3: In problem b. we are given $P\{X \le N\} \ge 0.90$.

Step 4: $-P\{X \le N\} \le -0.90$

$1 - P\{X \le N\} \le 1 - 0.90$

$P\{X > N\} = 1 - P\{X \le N\} \le 0.10$

➤d.
There is at least a 90% chance that the game will be complete by the 13th roll.

➤e.
There is less than a 10% chance that the game will need more than 13 rolls to complete the game.

13.
➤a.
Step 1: By definition $k \ge 2$.

Step 2: **F**: The event that a 5 is rolled.

Examples:

$k = 2$: $\{X = 2\} = \mathbf{F}_1 \cap \mathbf{F}_2$

$P\{\mathbf{F}_1 \cap \mathbf{F}_2\} = P\{\mathbf{F}_1) P(\mathbf{F}_2) = (1/6)(1/6) = (1/6)^2$

k = 3: $\{X = 3\} = (\mathbf{F}_1 \cap \mathbf{F}_2' \cap \mathbf{F}_3) \cup (\mathbf{F}_1' \cap \mathbf{F}_2 \cap \mathbf{F}_3)$

$P\{X = 3\} = P[(\mathbf{F}_1 \cap \mathbf{F}_2' \cap \mathbf{F}_3) \cup (\mathbf{F}_1' \cap \mathbf{F}_2 \cap \mathbf{F}_3)] = P[(\mathbf{F}_1 \cap \mathbf{F}_2' \cap \mathbf{F}_3) + P(\mathbf{F}_1' \cap \mathbf{F}_2 \cap \mathbf{F}_3) =$

$(1/6)(5/6)(1/6) + (5/6)(1/6)(1/6) = 2(5/6)(1/6)^2$

k = 4: $\{X = 4\} = (\mathbf{F}_1 \cap \mathbf{F}_2' \cap \mathbf{F}_3' \cap \mathbf{F}_4) \cup (\mathbf{F}_1' \cap \mathbf{F}_2 \cap \mathbf{F}_3' \cap \mathbf{F}_4) \cup (\mathbf{F}_1' \cap \mathbf{F}_2' \cap \mathbf{F}_3 \cap \mathbf{F}_4)$

$P\{X = 4\} = P(\mathbf{F}_1 \cap \mathbf{F}_2' \cap \mathbf{F}_3' \cap \mathbf{F}_4) + P(\mathbf{F}_1' \cap \mathbf{F}_2 \cap \mathbf{F}_3' \cap \mathbf{F}_4) + P(\mathbf{F}_1' \cap \mathbf{F}_2' \cap \mathbf{F}_3 \cap \mathbf{F}_4)$

$P\{X = 4\} = P(\mathbf{F}_1)P(\mathbf{F}_2')P(\mathbf{F}_3')P(\mathbf{F}_4) + P(\mathbf{F}_1')P(\mathbf{F}_2)P(\mathbf{F}_3')P(\mathbf{F}_4) + P(\mathbf{F}_1')P(\mathbf{F}_2')P(\mathbf{F}_3)P(\mathbf{F}_4) =$

$(1/6)(5/6)(5/6)(1/6) + (5/6)(1/6)(5/6)(1/6) + (5/6)(5/6)(1/6)(1/6) = 3(5/6)^2(1/6)^2$

The general form:

$\{X = k\} = (\mathbf{F}_1 \cap \mathbf{F}_2' \cap \mathbf{F}_3' \ldots \mathbf{F}_{k-1}' \cap \mathbf{F}_k) \cup (\mathbf{F}_1' \cap \mathbf{F}_2 \cap \mathbf{F}_3' \ldots \cap \mathbf{F}_{k-1}' \cap \mathbf{F}_k) \cup \ldots \cup (\mathbf{F}_1' \cap \mathbf{F}_2' \cap \mathbf{F}_3' \ldots \mathbf{F}_{k-1} \cap \mathbf{F}_k)$

$P\{X = k\} = P(\mathbf{F}_1 \cap \mathbf{F}_2' \cap \mathbf{F}_3' \ldots \mathbf{F}_{k-1}' \cap \mathbf{F}_k) + P(\mathbf{F}_1' \cap \mathbf{F}_2 \cap \mathbf{F}_3' \ldots \cap \mathbf{F}_{k-1}' \cap \mathbf{F}_k) + \ldots + P\mathbf{F}_1' \cap \mathbf{F}_2' \cap \mathbf{F}_3' \ldots \mathbf{F}_{k-1} \cap \mathbf{F}_k)$

$= (k - 1)(5/6)^{k-2}(1/6)^2$

➤b.

$\{4 < X \leq 10\} = \{X = 4\} \cup \{X = 5\} \cup \{X = 6\} \cup \{X = 7\} \cup \{X = 8\} \cup \{X = 9\} \cup \{X = 10\}$

$P\{4 < X \leq 10\} = P\{X = 4\} + P\{X = 5\} + P\{X = 6\} + P\{X = 7\} + P\{X = 8\} + P\{X = 9\} + P\{X = 10\} =$

$(3)(5/6)^2(1/6)^2 + (4)(5/6)^3(1/6)^2 + (5)(5/6)^4(1/6)^2 + (6)(5/6)^5(1/6)^2 + (7)(5/6)^6(1/6)^2 +$

$(8)(5/6)^7(1/6)^2 + (9)(5/6)^8(1/6)^2 \approx 0.44$

➤c.

Step 1: $\{X \geq 5\}' = \{X \leq 4\} = \{X = 2\} \cup \{X = 3\} \cup \{X = 4\}$

Step 2: $P\{X \geq 5\}' = P\{X \leq 4\} = P\{X = 2\} + P\{X = 3\} + P\{X = 4\} =$

$(1)(5/6)^0(1/6)^2 + (2)(5/6)^1(1/6)^2 + (3)(5/6)^2(1/6)^2 = \{1 + 10/6 + 75/36\}(1/36) =$

$\{36/36 + 60/36 + 75/36\}(1/36) \approx 0.13$

Step 3: $P\{X \geq 5\} = 1 - P\{X \leq 4\} \approx 0.87$

14.

➤a.

First game:

Assume a fair die is tossed 4 times.

X: The random variable equals the number of aces tossed.

$\mathbf{A}_k$: The event an ace is rolled on the kth toss.

$P(\mathbf{A}_k) = 1/6$

$\{X \geq 1\}$: The event of at least 1 ace occured.

$\{X \geq 1\}' = \{X = 0\} = A_1'\cap A_2'\cap A_3'\cap A_4'$

Because the events are independent:

$P\{X \geq 1\}' = P\{X = 0\} = P[\mathbf{A}_1'\cap\mathbf{A}_2'\cap\mathbf{A}_3'\cap\mathbf{A}_4'] = P(\mathbf{A}_1')P(\mathbf{A}_2')P(\mathbf{A}_3')P(\mathbf{A}_4') = (5/6)^4$

$P(X \geq 1) = 1 - P[(X \geq 1)'] = 1 - P(X = 0) = 1 - (1 - 1/6)^4 = 0.5177$

Second game:

Assume a pair of dice is tossed 24 times.

Y: The random variable equals the number of pair of aces tossed.

$\mathbf{D}_k$: The event a double ace is rolled on the kth toss.

$P(\mathbf{D}_k) = 1/36$

$\{Y \geq 1\}$: The event of at least 1 double ace occured.

$\{Y \geq 1\}' = \{Y = 0\} = \mathbf{D}_1'\cap\mathbf{D}_2'\cap\mathbf{D}_3'...\cap\mathbf{D}_{24}'$

Because the events are independent:

$P\{Y \geq 1\}' = P\{Y = 0\} = P[\mathbf{D}_1'\cap\mathbf{D}_2'\cap\mathbf{D}_3'...\cap\mathbf{D}_{24}'] = P(\mathbf{D}_1')P(\mathbf{D}_2')P(\mathbf{D}_3')...P(\mathbf{D}_{24}') = (1 - 35/36)^{24}$

$P(Y \geq 1) = 1 - P[(Y \geq 1)'] = 1 - P(Y = 0) = 1 - (35/36)^{24} = 0.4914$

➤b.

He reasoned the following way:

First game:

Step 1: The chance of tossing an ace is 1/6.

Step 2: There are 4 tosses.

The chance of at least 1 ace: 4(1/6) = 2/3.

Second game:

Step 1: The chance of tossing a pair of aces is 1/36

Step 2: There are 24 tosses.

The chance of at least 1 double aces: 24(1/36) = 2/3.

15.
➤a.
Step 1: $X = 1, Y = 1$

$Z = XY = (1)(1) = 1$

Step 2: $X = 1, Y = -1$

$Z = XY = (1)(-1)$

Step 3: $X = -1, Y = 1$

$Z = XY = (-1)(1) = 1$

Step 4: $X = -1, Y = -1$

$Z = XY = (-1)(-1) = 1$

Step 5: $\{Z = 1\} = [\{X = 1\}\cap\{Y = 1\}]\cup[\{X = -1\}\cap\{Y = -1\}]$

$P\{Z = 1\} = P\{[\{X = 1\}\cap\{Y = 1\}]\cup[\{X = -1\}\cap\{Y = -1\}]\} =$

$P[\{X = 1\}\cap\{Y = 1\}] + P[\{X = -1\}\cap\{Y = -1\}] = P\{X = 1\}P\{Y = 1\} + P\{X = -1\}P\{Y = -1\} =$

$(1/2)(1/2) + (1/2)(1/2) = 1/2$

Step 6: $\{Z = -1\} = [\{X = -1\}\cap\{Y = 1\}]\cup[\{X = 1\}\cap\{Y = -1\}]$

$P\{Z = -1\} = P\{[\{X = -1\}\cap\{Y = 1\}]\cup[\{X = 1\}\cap\{Y = -1\}]\} =$

$P[\{X = -1\}\cap\{Y = 1\}] + P[\{X = 1\}\cap\{Y = -1\}] = P\{X = -1\}P\{Y = 1\} + P\{X = 1\}P\{Y = -1\} =$

$(1/2)(1/2) + (1/2)(1/2) = 1/2$

Therefore,

$P\{Z = 1\} = 1/2$

$P\{Z = -1\} = 1/2$

➤b.
It is given that X and Y and independent.

We will now show that X and Z are independent.

Step 1: $\{X = 1\}\cap\{Z = 1\} = \{X = 1\}\cap\{Y = 1\}$

$P[\{X = 1\}\cap\{Z = 1\}] = P[\{X = 1\}\cap\{Y = 1\}] = P\{X = 1\}P\{Y = 1\} = (1/2)(1/2) =$

$P\{X =1 \}P\{Z = 1\}$

Step 2: $\{X = 1\}\cap\{Z = -1\} = \{X = 1\}\cap\{Y = -1\}$

$P[\{X = 1\}\cap\{Z = -1\}] = P[\{X = 1\}\cap\{Y = -1\}] = P\{X = 1\}P\{Y = -1\} = (1/2)(1/2) =$

$P\{X =1 \}P\{Z = -1\}$

Step 3: $\{X = -1\}\cap\{Z = 1\} = \{X = -1\}\cap\{Y = -1\}$

$P[\{X = -1\}\cap\{Z = 1\}] = P[\{X = -1\}\cap\{Y = -1\}] = P\{X = -1\}P\{Y = -1\} = (1/2)(1/2) =$

$P\{X =-1\}P\{Z = 1\}$

Step 4: $\{X = -1\}\cap\{Z = -1\} = \{X = -1\}\cap\{Y = 1\}$

$P[\{X = -1\}\cap\{Z = -1\}] = P[\{X = -1\}\cap\{Y = 1\}] = P\{X = -1\}P\{Y = 1\} = (1/2)(1/2) =$

$P\{X =-1\}P\{Z = -1\}$

Since Y has the same distribution as X, it followins that Y and Z are also independent.

Therefore X, Y, Z are piece-wise independent.

➤c.
To show they are not mutually independent, We only need to show

$P[Z = 1|\{X = -1\}\cap\{Y = 1\}] \neq P\{Z = 1\} = 1/2$.

From the definition of Z, $P[Z = 1|\{X = -1\}\cap\{Y = 1\}] = 0 \neq 1/2$.

16.
➤a.
We are given $m < n$.

From lesson 9, we define $\mathbf{A}\subseteq \mathbf{B}$ if $\mathbf{A}\cap\mathbf{B}' = \phi$

Step 1: $\{X < n\}' = \{X \geq n\}$

Step 2: $\{X < m\}\cap\{X < n\}' = \{X < m\}\cap\{X \geq n\} = \phi$

Therefore, $\{X<m\} \subseteq \{X<n\}$

➤b.
Step 1: $\{X > m\}' = \{X \leq m\}$

Step 2: $\{X > n\}\cap\{X > m\}' = \{X > n\}\cap\{X\leq m\} = \phi$

Therefore, $\{X > n\} \subseteq \{X> m\}$

17.
X: The random variable equals the number of rolls to terminate the game.

$\{X > 1\}$: The event it takes more than 1 roll to terminate the game.

$\{X >3\}$: The event it takes more than 3 rolls to terimnate the game.

$P[\{X>3\}|\{X>1\}]$: Given that an ace did not occur on the first toss, the probability that it will take more than 3 tosses.

Step 2: $P[\{X>3\}|\{X>1\}] = P[\{X>3\}\cap\{X>1\}]/P[\{X>1\}]$

$\{X>3\}\subseteq \{X >1\}$: See problem 16.b.

$\{X>3\}\cap\{X>1\} = \{X>3\}$: See axioms in lesson 9.

Therefore,

$P[\{X>3\}|\{X>1\}] = P[\{X>3\}\cap\{X>1\}]/P[\{X>1\}] = P[\{X>3\}]/P[\{X>1\}]=$

Step 3: By definition $P\{X = k\} = (5/6)^{k-1}(1/6)$ (See problem 13.)

Step 4: $P\{X>1\} = 1 - P\{X = 1\} = 1 - 1/6 = 5/6$

$P\{X>3\} = 1 - P\{X\leq 3\} = 1 - \{1/6 + (5/6)(1/6) + (5/6)^2(1/6)\} = 1 - (1/6)(1 + 5/36 + 25/36) =$

$1 - (1/6)(91/36) = 1 - 91/216 = 125/216$

Step 5: $P[\{X>3\}|\{X>1\}] = P[\{X>3\}]/P[\{X>1\}]= (125/216)(6/5) = 25/36$

18.

➤a.

$P\{X_j = k\} = q^k p, k = 0,1,2,...$

Step 1: $\{Z = n\} =$

$[\{X_1 = 0\}\cap\{X_2 = n\}]\cup[\{X_1 = 1\}\cap\{X_2 = n-1\}]\cup[\{X_1 = 2\}\cap\{X_2 = n - 2\}]\cup...\cup[\{X_1 =n\}\cap\{X_2 = 0\}]$

$P\{Z = n\} = P[\{X_1 = 0\}\cap\{X_2 = n\}]+P[\{X_1 = 1\}\cap\{X_2 = n-1\}]+P[\{X_1 = 2\}\cap\{X_2 = n - 2\}]+...+$

$P[\{X_1 =n\}\cap\{X_2 = 0\}] =$

$P\{X_1 = 0\}P\{X_2 = n\}+P\{X_1 = 1\}P\{X_2 = n-1\}+P\{X_1 = 2\}P\{X_2 = n - 2\}+...+P\{X_1 =n\}P\{X_2 = 0\} =$

$p(q^n p) + (qp)(q^{n-1}p) + (q^2p)(q^{n-2}p) + ... + (q^n p)(p) = p^2(n + 1)q^n$

Therefore, $p\{Z = n\} = p^2(n + 1)q^n$

➤b.

$P\{X_1 = k| Z = n\} = P[\{X_1 = k\}\cap\{Z = n\}]/P\{Z = n\}$, $k = 0,1,...,n$

$\{X_1 = k\}\cap\{Z = n\} = \{X_1 = k\}\cap\{X_2 = n - k\}$

Therefore, from a.

$P\{X_1 = k| Z = n\} = P[\{X_1 = k\}\cap\{Z = n\}]/P\{Z = n\} =P[\{X_1 = k\}\cap\{X_2 = n - k\}]/P\{Z = n\} =$

$(q^k p)(q^{n-k}p)/[p^2(n + 1)q^n] = 1/(n + 1)$

19.

➤a.

Since each card has equal probability of being in its proper position: $P\{X_k = 1\} = 1/N$

$P\{X_k = 0\} = 1 - 1/N = (N - 1)/N$

➤b.
$p(x_j,x_k) = P\{X_j = x_j; X_k = x_k\} = P[\{X_j = x_j\}\cap\{X_k = x_k\}]$ for all x_j, x_k.

$p(1,1) = P\{X_j = 1;X_k = 1\} = P[\{X_j = 1\}\cap\{X_k = 1\}] = P\{X_j = 1\}P\{X_k = 1|X_j = 1\} = (\frac{1}{N})(\frac{1}{N-1})$

$p(1,0) = P\{X_j = 1;X_k = 0\} = P[\{X_j = 1\}\cap\{X_k = 0\}] = P\{X_j = 1\}P\{X_k = 0|X_j = 1\} = (\frac{1}{N})(\frac{N-2}{N-1})$

$p(0,1) = P\{X_j = 0;X_k = 1\} = P[\{X_k = 1\}\cap\{X_i = 0\}] = (\frac{1}{N})(\frac{N-2}{N-1})$

$p(0,0) = P\{X_j = 0;X_k = 0\} = 1 - p(1,1) - p(1,0) - p(0,1) = \frac{N^2 - 3N + 3}{N(N-1)}$

➤c.
$P\{X_j = 1;X_k = 1\} = (\frac{1}{N})(\frac{1}{N-1})$

$P\{X_j = 1\} = 1/N$

$P\{X_k = 1\} = 1/N$

Therefore, $P\{X_j = 1;X_k = 1\} = (\frac{1}{N})(\frac{1}{N-1}) \neq (1/N)(1/N)$

➤d.
$p(1,0,0) = P\{X_j = 1;X_k = 0;X_r = 0\} = P\{X_j = 1\}P\{X_k = 0;X_r = 0| X_j = 1\}$

Step 1: From c. we know $P\{X_j = 1\} = 1/N$.

Step 2: $P\{X_k = 0;X_r = 0| X_j = 1\} = P\{X_k = 0;X_r = 0\}$ for N - 1.

Step 3: Since 1 card is in proper placement, replace N with N -1 in the formula for p(0,0) in c.:

$p(1,0,0) = P\{X_j = 1\}P\{X_k = 0;X_r = 0| X_j = 1\} = \frac{(N-1)^2 - 3(N-1) + 3}{N(N-1)(N-2)} =$

$\frac{N^2 - 5N + 7}{N(N-1)(N-2)}$

20.
➤a.
Each time heads is tossed $X_k = 1$. Each time tails is tossed $X_k = 0$. Therefore, S_N which is the sum of each X_k, will equal the number of heads tossed.

➤b.
Step 1: $P\{S_1 = 1; S_2 = 1\} = P[\{S_1 = 1\} \cap \{S_2 = 1\}] = P[\{X_1 = 1\} \cap \{X_2 = 0\}] = P\{X_1 = 1\}P\{X_2 = 0\}$ $= p(1 - p) = pq$

Step 2: $\{S_1 = 1\} = \{X_1 = 1\}$ and $\{S_2 = 1\} = [\{X_1 = 1\} \cap \{X_2 = 0\}] \cup [\{X_1 = 0\} \cap \{X_2 = 1\}]$

Step 3: $P\{S_1 = 1\} = P\{X_1 = 1\} = p$

and $P\{S_2 = 1\} = P[\{X_1 = 1\} \cap \{X_2 = 0\}] + P[\{X_1 = 0\} \cap \{X_2 = 1\}] =$

$P\{X_1 = 1\}P\{X_2 = 0\} + P\{X_1 = 0\}P\{X_2 = 1\} = pq + qp = 2pq$

Step 4: $P\{S_1 = 1; S_2 = 1\} = pq$

$P\{S_1 = 1\}P\{S_2 = 1\} = p(2pq)$

Therefore, $P\{S_1 = 1; S_2 = 1\} \neq P(S_1 = 1)P(S_2 = 1)$ which means the sequence S_k are NOT mutually independent.

➤c.
Step 1: $\{S_3 = 2\} =$

$\{X_1 = 0\} \cap \{X_2 = 1\} \cap \{X_3 = 1\} \cup \{X_1 = 1\} \cap \{X_2 = 0\} \cap \{X_3 = 1\} \cup \{X_1 = 1\} \cap \{X_2 = 1\} \cap \{X_3 = 0\}$

Step 2: $P\{S_3 = 2\} =$

$P[\{X_1 = 0\} \cap \{X_2 = 1\} \cap \{X_3 = 1\}] + P[\{X_1 = 1\} \cap \{X_2 = 0\} \cap \{X_3 = 1\}] +$

$P[\{X_1 = 1\} \cap \{X_2 = 1\} \cap \{X_3 = 0\}] = qpp + pqp + ppq = 3qp^2$

➤d.
Step 1:
$P\{S_2 = 2 | X_1 = 1\} = P[\{S_2 = 2\} \cap \{X_1 = 1\}]/P\{X_1 = 1\} = P[\{X_1 = 1\} \cap \{X_2 = 1\} \cap \{X_1\}]/P\{X_1 = 1\}$

$= P[\{X_1 = 1\} \cap \{X_2 = 1\}]/P\{X_1 = 1\} = P\{X_2 = 1\} = p$

Step 2:
$P\{X_1 = 1 | S_2 = 2\} = P[\{S_2 = 2\} \cap \{X_1 = 1\}]/P\{S_2 = 2\} = P[\{X_1 = 1\} \cap \{X_2 = 1\} \cap \{X_1\}]/P\{S_2 = 2\} =$

$P[\{X_1 = 1\} \cap \{X_2 = 1\}]/P\{S_2 = 2\} = P[\{X_1 = 1\} \cap \{X_2 = 1\}]/ P[\{X_1 = 1\} \cap \{X_2 = 1\}] = 1$

➤e.
Step 1:
$P\{S_1 = 1 | S_4 = 3\} = P\{(S_1 = 1) \cap (S_4 = 3)\}/P(S_4 = 3)$

$(S_1 = 1)\cap(S_4 = 3) =$

$\{(X_1 = 1)\cap(X_2 = 1)\cap(X_3 = 1)\cap(X_4 = 0)\}\cup\{(X_1 = 1)\cap(X_2 = 1)\cap(X_3 = 0)\cap(X_4 = 1)\}\cup$
$\{(X_1 = 1)\cap(X_2 = 0)\cap(X_3 = 1)\cap(X_4 = 1)\}$

$P[(S_1 = 1)\cap(S_4 = 3)] =$

$P\{(X_1 = 1)\cap(X_2 = 1)\cap(X_3 = 1)\cap(X_4 = 0)\} + P\{(X_1 = 1)\cap(X_2 = 1)\cap(X_3 = 0)\cap(X_4 = 1)\} +$

$P\{(X_1 = 1)\cap(X_2 = 0)\cap(X_3 = 1)\cap(X_4 = 1)\} = pppq + ppqp + pqpp = 3p^3q$

Step 2:
$P\{S_4 = 3\} = P[\{(X_1 = 1)\cap(X_2 = 1)\cap(X_3 = 1)\cap(X_4 = 0)\}\cup \{(X_1 = 1)\cap(X_2 = 1)\cap(X_3 = 0)\cap(X_4 = 1)\}\cup$

$\{(X_1 = 1)\cap(X_2 = 0)\cap(X_3 = 1)\cap(X_4 = 1)\}\cup\{(X_1 = 0)\cap(X_2 = 1)\cap(X_3 = 1)\cap(X_4 = 1)\}] =$

$P\{(X_1 = 1)\cap(X_2 = 1)\cap(X_3 = 1)\cap(X_4 = 0)\}+P\{(X_1 = 1)\cap(X_2 = 1)\cap(X_3 = 0)\cap(X_4 = 1)\}+$

$P\{(X_1 = 1)\cap(X_2 = 0)\cap(X_3 = 1)\cap(X_4 = 1)\}+P\{(X_1 = 0)\cap(X_2 = 1)\cap(X_3 = 1)\cap(X_4 = 1)\}] =$

$p^3q + p^2qp + pqpp + qp^3 = 4p^3q$

Step 3: $P\{S_1 = 1|S_4 = 3\} = P\{(S_1 = 1)\cap(S_4 = 3)\}/P(S_4 = 3) = (3p^3q)/(4p^3q) = 3/4$.

21.
Step 1: $(Y = y_1)\cup(Y = y_2)\cup \ldots \cup(Y = y_m) = \mathbf{S}$, Since Y equals each of y_k.

Step 2: $P(Y = y_1) + P(Y = y_2) + \ldots + P(Y = y_m) = P(\mathbf{S}) = 1$

Step 3: $p(x_, y_1) + p(x_, y_2) + \ldots + p(x_, y_{rm}) =$

$P(X = x)\cap(Y = y_1) + P(X = x)\cap(Y = y_2) + \ldots + P(X = x)\cap(Y = y_m) =$

$P\{(X = x)\cap[(Y = y_1)\cup(Y = y_2)\cup\ldots\cup(Y = y_m)]\} = P[(X = x)\cap\mathbf{S}] = P(X = x)$

22.
➤a.
Each time a head is rolled X_k will equal 1. Each time a tail is rolled X_k wil equal 0.
➤b.
Step 1: By Bayes, $P(N_1| S_1) = P(S_1|N_1)P(N_1)/P(S_1)$

$P(S_1|N_1) = P(X_1 = 1) = p$

$P(N_1) = 1/6$

Step 2: $S_1 = S_1 \cap[(N = 1)\cup(N = 2)\cup(N = 3)\cup(N = 4)\cup(N = 5)\cup(N = 6)] =$
$[S_1\cap(N = 1)]\cup[S_1\cap(N = 2)]\cup[S_1\cap(N = 3)]\cup[S_1\cap(N = 4)]\cup[S_1\cap(N = 5)]\cup[S_1\cap(N = 6)]$

Step 3: $S_1\cap(N = 1) = [(X_1 = 1)\cap(N = 1)]$

$S_1\cap(N = 2) = [(X_1= 1)\cap(X_2 = 0)\cap(N = 2)]\cup[(X_1 = 0)\cap(X_2 = 1)\cap(N = 2)]$

$S_1\cap(N = 3) =$
$[(X_1= 1)\cap(X_2 = 0)\cap(X_3 = 0)\cap(N = 3)]\cup[(X_1= 0)\cap(X_2 = 1)\cap(X_3 = 0)\cap(N = 3)]\cup$
$[(X_1 = 0)\cap(X_2 = 0)\cap(X_3 = 1)\cap(N = 3)]$

$S_1\cap(N = 4) =$
$[(X_1= 1)\cap(X_2 = 0)\cap(X_3 = 0)\cap(X_4 = 0)\cap(N = 4)]\cup$
$[(X_1= 0)\cap(X_2 = 1)\cap(X_3 = 0)\cap(X_4 =)\cap(N = 4)]\cup$
$[(X_1= 0)\cap(X_2 = 0)\cap(X_3 = 1)\cap(X_4 = 0)\cap(N = 4)]\cup$
$[(X_1= 0)\cap(X_2 = 0)\cap(X_3 = 0)\cap(X_4 = 1)(N = 4)]$

$S_1\cap(N = 5) =$
$[(X_1= 1)\cap(X_2 = 0)\cap(X_3 = 0)\cap(X_4 = 0)\cap(X_5 = 0)\cap(N = 5)]\cup$
$[(X_1= 0)\cap(X_2 = 1)\cap(X_3 = 0)\cap(X_4 = 0)\cap(X_5 = 0)\cap(N = 5)]\cup$
$[(X_1= 0)\cap(X_2 = 0)\cap(X_3 = 1)\cap(X_4 = 0)\cap(X_5 = 0)\cap(N = 5)]\cup$
$[(X_1= 0)\cap(X_2 = 0)\cap(X_3 = 0)\cap(X_4 = 1)\cap(X_5 = 0)\cap(N = 5)]\cup$
$[(X_1= 0)\cap(X_2 = 0)\cap(X_3 = 0)\cap(X_4 = 0)\cap(X_5 = 1)\cap(N = 5)]$

$S_1\cap(N = 6) =$
$[(X_1= 1)\cap(X_2 = 0)\cap(X_3 = 0)\cap(X_4 = 0)\cap(X_5 = 0)\cap(X_6 = 1)\cap(N = 5)]\cup$
$[(X_1= 0)\cap(X_2 = 1)\cap(X_3 = 0)\cap(X_4 = 0)\cap(X_5 = 0)\cap(X_6 = 1)\cap(N = 5)]\cup$
$[(X_1= 0)\cap(X_2 = 0)\cap(X_3 = 1)\cap(X_4 = 0)\cap(X_5 = 0)\cap(X_6 = 1)\cap(N = 5)]\cup$
$[(X_1= 0)\cap(X_2 = 0)\cap(X_3 = 0)\cap(X_4 = 1)\cap(X_5 = 0)\cap(X_6 = 1)\cap(N = 5)]\cup$
$[(X_1= 0)\cap(X_2 = 0)\cap(X_3 = 0)\cap(X_4 = 0)\cap(X_5 = 1)\cap(X_6 = 1)\cap(N = 5)]\cup$
$[(X_1= 0)\cap(X_2 = 0)\cap(X_3 = 0)\cap(X_4 = 0)\cap(X_5 = 1)\cap(X_6 = 1)\cap(N = 5)]$

Step 4:
$P[S_1\cap(N = 1)] = P[(X_1 = 1)\cap(N = 1)] = P[(X_1 = 1)|(N = 1)]P(N = 1)] = p(1/6)$

$P[S_1 \cap (N = 2)] = P[(X_1 = 1) \cap (X_2 = 0) \cap (N = 2) \cup (X_1 = 0)(X_2 = 1) \cap (N = 2)] =$
$P[(X_1 = 1) \cap (X_2 = 0) | (N = 2)]P(N = 2) + P[(X_1 = 0) \cap (X_2 = 1) | (N = 2)]P(N = 2) =$
$pq(1/6) + pq(1/6) = 2pq(1/6)$

$P[S_1 \cap (N = 3)] =$
$P[(X_1 = 1) \cap (X_2 = 0) \cap (X_3 = 0) \cap (N = 3)] + P[(X_1 = 0) \cap (X_2 = 1) \cap (X_3 = 0) \cap (N = 3)] +$
$P[(X_1 = 0) \cap (X_2 = 0) \cap (X_3 = 1) \cap (N = 3)] =$

$P[(X_1 = 1) \cap (X_2 = 0) \cap (X_3 = 0) | (N = 3)]P(N = 3) + P[(X_1 = 0) \cap (X_2 = 1) \cap (X_3 = 0) | (N = 3)]P(N = 3)$

$+ P[(X_1 = 0) \cap (X_2 = 0) \cap (X_3 = 1) | (N = 3)]P(N = 3) = 3pq^2(1/6)$

$P[S_1 \cap (N = 4)] =$
$P[(X_1 = 1) \cap (X_2 = 0) \cap (X_3 = 0) \cap (X_4 = 0) \cap (N = 4)] +$
$P[(X_1 = 0) \cap (X_2 = 1) \cap (X_3 = 0) \cap (X_4 =) \cap (N = 4)] +$
$P[(X_1 = 0) \cap (X_2 = 0) \cap (X_3 = 1) \cap (X_4 = 0) \cap (N = 4)] +$
$P[(X_1 = 0) \cap (X_2 = 0) \cap (X_3 = 0) \cap (X_4 = 1)(N = 4)] = 4qp^3(1/6)$

$P[S_1 \cap (N = 5)] =$
$P[(X_1 = 1) \cap (X_2 = 0) \cap (X_3 = 0) \cap (X_4 = 0) \cap (X_5 = 0) \cap (N = 5)] +$
$P[(X_1 = 0) \cap (X_2 = 1) \cap (X_3 = 0) \cap (X_4 = 0) \cap (X_5 = 0) \cap (N = 5)] +$
$P[(X_1 = 0) \cap (X_2 = 0) \cap (X_3 = 1) \cap (X_4 = 0) \cap (X_5 = 0) \cap (N = 5)] +$
$P[(X_1 = 0) \cap (X_2 = 0) \cap (X_3 = 0) \cap (X_4 = 1) \cap (X_5 = 0) \cap (N = 5)] +$
$P[(X_1 = 0) \cap (X_2 = 0) \cap (X_3 = 0) \cap (X_4 = 0) \cap (X_5 = 1) \cap (N = 5)] = 5pq^4(1/6)$

$P[S_1 \cap (N = 6)] =$
$P[(X_1 = 1) \cap (X_2 = 0) \cap (X_3 = 0) \cap (X_4 = 0) \cap (X_5 = 0) \cap (X_6 = 1) \cap (N = 5)] +$
$P[(X_1 = 0) \cap (X_2 = 1) \cap (X_3 = 0) \cap (X_4 = 0) \cap (X_5 = 0) \cap (X_6 = 1) \cap (N = 5)] +$
$P[(X_1 = 0) \cap (X_2 = 0) \cap (X_3 = 1) \cap (X_4 = 0) \cap (X_5 = 0) \cap (X_6 = 1) \cap (N = 5)] +$
$P[(X_1 = 0) \cap (X_2 = 0) \cap (X_3 = 0) \cap (X_4 = 1) \cap (X_5 = 0) \cap (X_6 = 1) \cap (N = 5)] +$
$P[(X_1 = 0) \cap (X_2 = 0) \cap (X_3 = 0) \cap (X_4 = 0) \cap (X_5 = 1) \cap (X_6 = 1) \cap (N = 5)] +$
$P[(X_1 = 0) \cap (X_2 = 0) \cap (X_3 = 0) \cap (X_4 = 0) \cap (X_5 = 1) \cap (X_6 = 1) \cap (N = 5)] = 6pq^5(1/6)$

Step 5:
$P(S_1) = P\{[S_1 \cap (N = 1)] \cup [S_1 \cap (N = 2)] \cup [S_1 \cap (N = 3)] \cup [S_1 \cap (N = 4)] \cup [S_1 \cap (N = 5)] \cup [S_1 \cap (N = 6)]\} =$

$P\{[S_1 \cap (N = 1)] + P[S_1 \cap (N = 2)] + P[S_1 \cap (N = 3)] + P[S_1 \cap (N = 4)] + P[S_1 \cap (N = 5)] + P[S_1 \cap (N = 6)] =$

$(p + 2pq + 3pq^2 + 4pq^3 + 5pq^4 + 6pq^5)(1/6)$

Step 6: $P(N_1 | S_1) = P(S_1|N_1)P(N_1)/P(S_1) = \dfrac{\frac{p}{6}}{\frac{p+2pq+3pq^2+4pq^3+5pq^4+6pq^5}{6}} =$

$$\frac{1}{1+2q+3q^2+4q^3+5q^4+6q^5}$$

23.
Step 1: $(S = n) = \{(X = n - 1) \cap (Y = 1)\} \cup \{(X = n - 2) \cap (Y = 2)\} \cup \ldots \cup (X = n - k) \cap (Y = k) \cup$

$\{(X = 1) \cap (Y = n - 1)\}$

Step 2 :
$P(S = n) = P\{(X = n - 1) \cap (Y = 1)\} + P\{(X = n - 2) \cap (Y = 2)\} + \ldots + P\{(X = n - k) \cap (Y = k)\} +$

$P\{(X = 1) \cap (Y = n - 1)\} = (q^{n-2}p)p + (q^{n-3}p)(pq) + \ldots + (q^{n-k-1}p)(q^{k-1}p) + \ldots + p(q^{n-2}p) =$

$q^{n-2}p^2 + q^{n-2}p^2 + \ldots + q^{n-2}p^2 + \ldots + q^{n-2}p^2) = (n - 1)q^{n-2}p^2$

24.
$P\{X = x_1\} + P\{X = x_2\} + \ldots + P\{X = x_N\} = 1$

$P\{X = x_1|\mathbf{A}\} + P\{X = x_2|\mathbf{A}\} + \ldots + P\{X = x_N|\mathbf{A}\} =$

$P[(X = x_1) \cap \mathbf{A}]/P(\mathbf{A}) + P[(X = {}_{X2}) \cap \mathbf{A}]/P(\mathbf{A}) + \ldots + P[(X = x_N) \cap \mathbf{A}]/P(\mathbf{A}) =$

$\{P[(X = x_1) \cap \mathbf{A}] + P[(X = {}_{X2}) \cap \mathbf{A}] + \ldots + P[(X = x_N) \cap \mathbf{A}]\}/P(\mathbf{A}) =$

$P\{[(X = x_1) \cap \mathbf{A}\ ] \cup (X = {}_{X2}) \cap \mathbf{A}\ \ldots\ \cup (X = x_N) \cap \mathbf{A}\}/P(\mathbf{A}) =$

$P\{[(X = x_1) \cup (X = {}_{X2}) \cup \ldots \cup (X = x_N)] \cap \mathbf{A}\}/P(\mathbf{A}) =$

$P(\mathbf{S} \cap \mathbf{A})/P(\mathbf{A}) = P(\mathbf{A})/P(\mathbf{A}) = 1.$

25.
H: A head is tossed.
X: Random variable equals the number of tosses to terminate the game.

Step 1: $P(X = k|\mathbf{H}) = P(\mathbf{H}|X = k)P(X = k)/P(\mathbf{H})$

$\mathbf{H}' = \mathbf{T}_1 \cap \mathbf{T}_2 \cap \mathbf{T}_3 \cap \mathbf{T}_4 \cap \mathbf{T}_5$

$P(\mathbf{H}') = P(\mathbf{T}_1 \cap \mathbf{T}_2 \cap \mathbf{T}_3 \cap \mathbf{T}_4 \cap \mathbf{T}_5) = P(\mathbf{T}_1)P(\mathbf{T}_2)P(\mathbf{T}_3)P(\mathbf{T}_4)P(\mathbf{T}_5) = (1/2)^5 = 1/32$

$P(\mathbf{H}) = 1 - P(\mathbf{H}') = 1 - 1/32 = 31/32$

$P(X = k|\mathbf{H}) = P(\mathbf{H}|X = k)P(X = k)/P(\mathbf{H}) = P(\mathbf{H}|X = k)P(X = k)(32/31)$

Step 2: $P(\mathbf{H}|X = k) = 1$, for $(k = 1,2,3,4)$

$P(X = k|\mathbf{H}) = P(\mathbf{H}|X = k)P(X = k)/P(\mathbf{H}) = P(X = k)(32/31)$ for $(k = 1,2,3,4)$

Step 3: $(X = 1) = \mathbf{H}_1$

$(X = 2) = \mathbf{H}_1' \cap \mathbf{H}_2$

$(X = 3) = \mathbf{H}_1' \cap \mathbf{H}_2' \cap \mathbf{H}_3$

$(X = 4) = \mathbf{H}_1' \cap \mathbf{H}_2' \cap \mathbf{H}_3' \cap \mathbf{H}_4$

$P(X = 1) = P(\mathbf{H}_1) = 1/2$

$P(X = 2) = P(\mathbf{H}_1' \cap \mathbf{H}_2) = P(\mathbf{H}_1')P(\mathbf{H}_2) = (1/2)(1/2) = 1/4$

$P(X = 3) = P(\mathbf{H}_1' \cap \mathbf{H}_2' \cap \mathbf{H}_3) = P(\mathbf{H}_1')P(\mathbf{H}_2')P(\mathbf{H}_3) = (1/2)(1/2)(1/2) = 1/8$

$P(X = 4) = P(\mathbf{H}_1' \cap \mathbf{H}_2' \cap \mathbf{H}_3' \cap \mathbf{H}_4) = P(\mathbf{H}_1')P(\mathbf{H}_2')P(\mathbf{H}_3')P(\mathbf{H}_4) = (1/2)(1/2)(1/2)(1/2) = 1/16$

Step 4: From step 2, we have $P(X = k|\mathbf{H}) = P(X = k)(32/31)$ for $(k = 1,2,3,4)$

$P(X = 1|\mathbf{H}) = P(X = 1)(32/31) = (1/2)(32/31) = 16/31$

$P(X = 2|\mathbf{H}) = P(X = 2)(32/31) = (1/4)(32/31) = 8/31$

$P(X = 3|\mathbf{H}) = P(X = 3)(32/31) = (1/8)(32/31) = 4/31$

$P(X = 4|\mathbf{H}) = P(X = 4)(32/31) = (1/16)(32/31) = 2/31$

Step 5: From problem 24, we have

$P(X = 1|\mathbf{H}) + P(X = 2|\mathbf{H}) + P(X = 3|\mathbf{H}) + P(X = 4|\mathbf{H}) + P(X = 5|\mathbf{H}) = 1.$

$16/31 + 8/31 + 4/31 + 2/31 + P(X = 5|\mathbf{H}) = 1$

30/31 + P(X = 5|**H**) = 1

P(X = 5|**H**) = 1 - 30/31 = 1/31

26.
$P(X = k) = pq^{k-1}$ (k = 1,2,...).

$P(X > n) = 1 - P(X \le n) = 1 - [P(X = 1) + P(X = 2) + ... + P(X = n)] = 1 - [p + pq + ... + pq^{n-1}] =$

$1 - p[1 + q + ... + q^{n-1}] = 1 - p[(1 - q^n)/(1 - q) = 1 - p[(1 - q^n)/p = 1 - (1 - q^n) = q^n$.

$P(X = n + k| X > n) = P[(X = n + k)\cap(X > n)]/P(X > n) = P(X = n + k)/q^n = (pq^{n+k-1})/q^n = pq^{k-1} =$

P(X = k).

$P(X = n + k| X > n) = P[(X = n + k)\cap(X > n)]/P(X > n) = P(X = n + k)/P(X > n)$

27.
$P(X = n + k| X > n) = P[(X = n + k)\cap(X > n)]/P(X > n) = P[(X = n + k)]/P(X > n) = P(X = k)$

$P(X = n + k) = P(X > n)P(X = k)$, k,n $\ge$ 1.

Assign P(X = 1) = p, and for n = 1, P(X > 1) = 1 - P(X = 1) = 1 - p = q.

Let k = 1, n = 1

P(X = 1 + 1 = 2) = P(X > 1)P(X = 1) = qp = pq

k = 1, n = 2

$P(X > 2) = 1 - P(X = 1) - P(X = 2) = 1 - p - pq = 1 - (1 - q) - (1 - q)q = q^2$

$P(X = 1 + 2 = 3) = P(X > 2)P(X = 1) = q^2p = pq^2$

Continuing by induction we have

$P(X = n + 1) = P(X > n)P(X = 1) = pq^n$

28.
This means that the only discrete r.v. X with the property P(X = n + k| X > n) = P(X = k) for k,n $\ge$ 1 is the geometric distribution.

29. The condition P(X = n + k| X > n) = P(X = k) is called 'the lack of memory' property. Explain in your own words why this is so.

This means that the random variable X ′s outcome is independent of the time that it starts. This means we can write P(X = n + k - n | X > n - n) = P(X = k| X >0) = P(X = k)

30.

Since he will play at most three games, let $\mathbf{W}_1$ be the event that he wins on the first game. Let $\mathbf{W}_2$ be the event that he wins on the second game. Let $\mathbf{W}_3$ be the event that he wins on the third game. All these events are independent. Let X represent the total amount he wins or loses. There are four possible values that X will equal:

Case 1: he wins all three games: $\mathbf{W}_1 \cap \mathbf{W}_2 \cap \mathbf{W}_3 = \{X = \$300\}$.

Case 2: he wins the first two games and loses the last game: $\mathbf{W}_1 \cap \mathbf{W}_2 \cap \mathbf{W}_3' = \{ X = \$90\}$

Case 3: he wins the first game and losses the second: $\mathbf{W}_1 \cap \mathbf{W}_2' = \{X = -\$10\}$.

Case 4: He loses the first game: $\mathbf{W}_1' = \{X = -\$110\}$.

From these cases we can compute:

$P\{X = \$300\} = P(\mathbf{W}_1 \cap \mathbf{W}_2 \cap \mathbf{W}_3) = (0.6)(0.6)(0.6) = 0.216$

$P\{X = \$90\} = P(\mathbf{W}_1 \cap \mathbf{W}_2 \cap \mathbf{W}_3') = (0.6)(0.6)(0.4) = 0.144$

$P\{X = -\$10\} = P(\mathbf{W}_1 \cap \mathbf{W}_2') = (0.6)(0.4) = 0.24$

$P\{X = -\$110\} = P(\mathbf{W}_1') = 0.4$

In table form:

x	**P{X = x}**
-$110	0.400
-$10	0.240
$90	0.144
$300	0.216
	Total 1