PROBABILITY THEORY
Lesson 12
Conditional Probability

12.1- What is Conditional Probability?

12.1 - Problem 1:

Step 1 : **A**: The event Mrs. Jones is arrested for the murder.

G: The event that she is found guilty.

A∩G: The event that she is arrested for the murder and found guilty.

Step 2: P(**A**) = 0.75

Step 3: P(**A**∩**G**) = 0.20

Step 4: $P(\mathbf{G}|\mathbf{A}) = \dfrac{P(\mathbf{A}\cap\mathbf{G})}{P(\mathbf{A})} = \dfrac{0.20}{0.75} = \dfrac{4}{15}$.

12.1- Problem 2:

➤(a).

Since a diamond is drawn and there are 3 face cards that are diamonds and 13 diamonds, the probability is P(**F**|**D**) = 3/13 .

➤(b).

Since a diamond is not drawn and there are 9 face cards that are not diamonds and 39 cards that are not diamonds, the probability is P(**F|D**′) = 9/39 = 3/13 .

12.1 - Problem 3:

➤(a).

Step 1: There are 52 cards and 13 diamonds.

Step 2: If the first card drawn is a diamond, after the first drawing there are 51 cards left and 12 diamonds left for the second drawing. Therefore, $P(\mathbf{D}_2|\ \mathbf{D}_1) = 12/51$.

➤(b).

Step 1: There are 52 cards and 13 diamonds.

Step 2: If the first card drawn is a diamond, after the first drawing there are 51 cards left and 39 non - diamonds left for the second drawing. Therefore, $P(\mathbf{D}_2'|\ \mathbf{D}_1) = 39/51$.

➤(c).
Step 1: There are 52 cards and 13 diamonds.

Step 2: If the first card drawn is not a diamond, after the first drawing there are 51 cards left and 13 diamonds left for the second drawing. Therefore,

$P(\mathbf{D}_2|\mathbf{D}_1') = 13/51$

➤(d).
Step 1: There are 52 cards and 13 diamonds.

Step 2: If the first card drawn is not a diamond, after the first drawing there are 51 cards left and 38 non - diamonds left for the second drawing. Therefore, $P(\mathbf{D}_2'|\mathbf{D}_1') = 38/51$.

➤(e).
Step 1: There are 52 cards and 13 diamonds.

Step 2: If the first card drawn is a diamond, after the first drawing there are 51 cards left and 13 hearts left for the second drawing. Therefore, $P(\mathbf{H}_2|\mathbf{D}_1) = 13/51$.

12.2 - A Formula for P(E∩B).

12. 2 - Problem 1:
Step 1:
$\mathbf{K}_1$: The event that the first card drawn is a king.

$\mathbf{A}_2$: The event that the second card drawn is a ace.

$\mathbf{Q}_2$: The event that the second card drawn is a queen.

E: The event that the first card is a king and the second card is an ace or queen: $\mathbf{E} = \mathbf{K}_1\cap(\mathbf{A}_2\cup\mathbf{Q}_2)$.

Step 2: The probability that the first card drawn is a king: $P(\mathbf{K}_1) = 4/52$.

Step 3: Given that the first card is a king, the probability that the second card is a ace or queen:

$P(\mathbf{A}_2\cup\mathbf{Q}_2|\mathbf{K}_1) = 8/51$ (there are 4 aces and 4 queens to be selected)

Step 4: Therefore, $P(\mathbf{E}) = P[\mathbf{K}_1\cap(\mathbf{A}_2\cup\mathbf{Q}_2)] = (\frac{4}{52})(\frac{8}{51}) = \frac{32}{2652} = \frac{8}{663}$.

12.2 - Problem 2:
➤(a).
Step 1: For the first drawing, there are 52 cards and 13 clubs. Therefore, the probability of drawing

a club is 13/52.

Step 2: For the second drawing there are 51 cards left and 12 clubs left. Therefore, the probability of drawing a club on the first drawing and a club on the second drawing is

(13/52)(12/51).

Step 3: For the third drawing there are 50 cards left and 11 clubs left. Therefore, the probability of drawing club on the first drawing, a club on the second drawing and a club on the third drawing is

(13/52)(12/51)(11/50) = 1716/132600.

➤(b).
Step 1: Let **E** = the event that at least one club is drawn.

Step 2: **E**′ = the event that no clubs are drawn.

Step 1: For the first drawing, there are 52 cards and 39 non- clubs. Therefore, the probability of not drawing a club is 39/52.

Step 2: For the second drawing there are 51 cards left and 38 clubs left. Therefore, the probability of not drawing a club on the first drawing and not drawing a club on the second drawing is

(39/52)(38/51).

Step 3: For the third drawing there are 50 cards left and 37 non-clubs left. Therefore, the probability of not drawing club on the first drawing, not drawing a club on the second drawing and not drawing a club on the third drawing is

P(**E**′) = (39/52)(38/51)(37/50) = 54834/132600 .

P(**E**) = 1 - P(**E**′) = 77766/ 132600

12.2 - Problem 3:

Step 1:
$\mathbf{R_1}$: The event that the marble drawn is red.
$\mathbf{B_2}$: The event that the second marble drawn is blue.
$\mathbf{R_3}$**:** The event that the third marble drawn is red.

E: The event that the marble drawn is red, the second marble drawn is blue and the third marble drawn is red:

$\mathbf{E} = \mathbf{R}_1 \cap \mathbf{B}_2 \cap \mathbf{R}_3$.

Step 2: The probability that the first marble drawn is a red:

$P(\mathbf{R}_1) = 25/50$.

Step 3: Given that the first marble drawn is red, the probability that the second marble is blue:

$P(\mathbf{B}_2 | \mathbf{R}_1) = 15/49$.

Step 4: Given that the first marble drawn is red, the second marble is blue , the probability that the third marble selected is red:

$P(\mathbf{R}_3 | \mathbf{R}_1 \cap \mathbf{B}_2) = 24/48$.

Step 5: $P(\mathbf{E}) = P(\mathbf{R}_1 \cap \mathbf{B}_2 \cap \mathbf{R}_3) = (\frac{25}{50})(\frac{15}{49})(\frac{24}{48}) = \frac{15}{196}$

12.3 - Writing the Event E in Terms of Other Events.

12.3 - Problem 1:

Step 1:

$\mathbf{U}_1$: The event urn 1 is selected.

$\mathbf{U}_2$: The event urn 2 is selected.

W: The event a white marble is selected.

R: The event a red marble is selected.

E: The event a red or white marble is selected: $\mathbf{E} = \mathbf{W} \cup \mathbf{R}$.

Step 2:The event **E** can occur as follows:

urn 1 is selected **and** a red or white marble is selected: $\mathbf{U}_1 \cap (\mathbf{W} \cup \mathbf{R})$

or

urn 2 is selected **and** a red or white marble is selected: $\mathbf{U}_2 \cap (\mathbf{W} \cup \mathbf{R})$.

Step 3: Therefore, $\mathbf{E} = [\mathbf{U}_1 \cap (\mathbf{W} \cup \mathbf{R})] \cup [\mathbf{U}_2 \cap (\mathbf{W} \cup \mathbf{R})]$

Step 4: $P(\mathbf{E}) = P([\mathbf{U}_1 \cap (\mathbf{W} \cup \mathbf{R})] \cup [\mathbf{U}_2 \cap (\mathbf{W} \cup \mathbf{R})]) = P([\mathbf{U}_1 \cap (\mathbf{W} \cup \mathbf{R})] + P[\mathbf{U}_2 \cap (\mathbf{W} \cup \mathbf{R})])$

Step 5: From the formula $P(\mathbf{B} \cap \mathbf{A}) = P(\mathbf{B} | \mathbf{A})P(\mathbf{A})$, we have

$P(\mathbf{E}) = P[\mathbf{U}_1 \cap (\mathbf{W} \cup \mathbf{R})] + P[\mathbf{U}_2 \cap (\mathbf{W} \cup \mathbf{R})] = P[(\mathbf{W} \cup \mathbf{R}) | \mathbf{U}_1]P(\mathbf{U}_1) + P[(\mathbf{W} \cup \mathbf{R}) | \mathbf{U}_2]P(\mathbf{U}_1)$

Step 6: $P[(\mathbf{W} \cup \mathbf{R}) | \mathbf{U}_1] = (10 + 15)/30 = 25/30$.

$P[(\mathbf{W} \cup \mathbf{R}) | \mathbf{U}_2] = (10 + 10)/30 = 20/30$

Step 7: If a king is selected, then urn 1 is selected. Therefore, $P(\mathbf{U_1}) = 4/52$.

If a king is not selected, the urn 2 is selected. Therefore, $P(\mathbf{U_2}) = 48/52$.

Step 6: From step 5 we have

$$P(\mathbf{E}) = P[(\mathbf{W} \cup \mathbf{R}) | \mathbf{U}_1]P(\mathbf{U}_1) + P[(\mathbf{W} \cup \mathbf{R}) | \mathbf{U}_2]P(\mathbf{U}_2) = \left(\frac{25}{30}\right)\left(\frac{4}{52}\right) + \left(\frac{20}{30}\right)\left(\frac{48}{52}\right) = \frac{53}{78}$$

12.3 - Problem 2:

➤(a).

The chance of drawing a face card on the second drawing depends on whether a face card had been drawn on the first drawing.

$\mathbf{F}_1$: the event that a face card is drawn on the first draw.

$\mathbf{F_2}$: the event that a face card is drawn on the second drawing.

$\mathbf{F_1} \cap \mathbf{F_2}$: the event that a face card is drawn on the first **AND** second drawing.

$\mathbf{F_1}' \cap \mathbf{F_2}$: the event that a face card is **NOT** drawn on the first **AND** a face card is drawn on the second drawing.

$\mathbf{F}_2 = (\mathbf{F}_1 \cap \mathbf{F}_2) \cup (\mathbf{F}_1' \cap \mathbf{F}_2)$: a face card is drawn on first and on the second or a face card is not drawn on the first but a face card is drawn on the second.

$$P(\mathbf{F}_2) = P(\mathbf{F_1} \cap \mathbf{F_2}) + P(\mathbf{F_1}' \cap \mathbf{F_2}) = \left(\frac{12}{52}\right)\left(\frac{11}{51}\right) + \left(\frac{40}{52}\right)\left(\frac{12}{51}\right) = \frac{663}{2652} = \frac{12}{52}$$

➤(b).

$\mathbf{F}_1$: the event that a face card is drawn on the first draw.

$\mathbf{F_2}$: the event that a face card is drawn on the second draw.

E: the event that a face card is drawn on the first or second drawing.

$\mathbf{E} = \mathbf{F_1} \cup \mathbf{F_2}$

$$P(\mathbf{E}) = P(\mathbf{F_1}\cup\mathbf{F_2}) = P(\mathbf{F_1}) + P(\mathbf{F_2}) - P(\mathbf{F_1}\cap\mathbf{F_2}) = \frac{12}{52} + \frac{12}{52} - (\frac{12}{52})(\frac{11}{51}) = \frac{1092}{2652}$$

➤(c).
E: the event that a face card and an ace are drawn.

Since order is not required we must consider all possibilities: $\mathbf{E} = (\mathbf{F}_1\cap\mathbf{A}_2)\cup(\mathbf{A}_1\cap\mathbf{F}_2)$.

$$P(\mathbf{E}) = P(\mathbf{F}_1\cap\mathbf{A}_2) + P(\mathbf{A}_1\cap\mathbf{F}_2) = P(\mathbf{F}_1)P(\mathbf{A}_2|\mathbf{F}_1) + P(\mathbf{A}_1)P(\mathbf{F}_2|\mathbf{A}_1) =$$

$$(\frac{12}{52})(\frac{4}{51}) + (\frac{12}{52})(\frac{4}{51}) = \frac{96}{2652}$$

➤(d).
E: at least 1 face card or ace is drawn.

$\mathbf{E} = (\mathbf{F}_1\cup\mathbf{A}_2)\cup(\mathbf{A}_1\cup\mathbf{F}_2)$

$\mathbf{E}'$: no face card and no ace are drawn.

$\mathbf{E}' = (\mathbf{F}_1'\cap\mathbf{A}_2')\cap(\mathbf{A}_1'\cap\mathbf{D}_2') = (\mathbf{D}_1'\cap\mathbf{C}_1')\cap(\mathbf{D}_2'\cap\mathbf{C}_2')$

$$P(\mathbf{E}') = P[(\mathbf{F}_1'\cap\mathbf{A}_1')\cap(\mathbf{A}_2'\cap\mathbf{F}_2')] = P(\mathbf{F}_1'\cap\mathbf{A}_1')P[(\mathbf{F}_2'\cap\mathbf{A}_2')|(\mathbf{F}_1'\cap\mathbf{A}_1')] =$$

$$(\frac{36}{52})(\frac{35}{51}) = \frac{1260}{2652}$$

$$P(\mathbf{E}) = 1 - P(\mathbf{E}') = 1 - \frac{1260}{2652} = \frac{1392}{2652}$$

12.3 - Problem 3:
➤(a).
$\mathbf{D_1}$: the event that a diamond is drawn on the first draw.
$\mathbf{F_1}$**:** the event that a face card is drawn on the first draw.
$\mathbf{F_2}$: the event that a face card is drawn on the second draw.
$\mathbf{E} = \mathbf{D_1}\cap\mathbf{F_2}$: the event that a diamond is drawn on the first drawing and a face card on the second drawing.

$\mathbf{D_1} = (\mathbf{D}_1\cap\mathbf{F}_1)\cup(\mathbf{D}_1\cap\mathbf{F}_1')$

$\mathbf{E} = \mathbf{D_1}\cap\mathbf{F_2} = [(\mathbf{D}_1\cap\mathbf{F}_1)\cup(\mathbf{D}_1\cap\mathbf{F}_1')]\cap\mathbf{F}_2 = [(\mathbf{D}_1\cap\mathbf{F}_1)\cap\mathbf{F}_2]\cup[(\mathbf{D}_1\cap\mathbf{F}_1')\cap\mathbf{F}_2]$

$$P(\mathbf{E}) = P(\mathbf{D}_1\cap\mathbf{F}_2) = P([(\mathbf{D}_1\cap\mathbf{F}_1)\cap\mathbf{F}_2]\cup[(\mathbf{D}_1\cap\mathbf{F}_1')\cap\mathbf{F}_2]) = P[(\mathbf{D}_1\cap\mathbf{F}_1)\cap\mathbf{F}_2] + P[(\mathbf{D}_1\cap\mathbf{F}_1')\cap\mathbf{F}_2] =$$

$P(\mathbf{D_1}\cap\mathbf{F_1})P[\mathbf{F_2}|(\mathbf{D_1}\cap\mathbf{F_1})] + P(\mathbf{D_1}\cap\mathbf{F_1}')P[\mathbf{F_2}|(\mathbf{D_1}\cap\mathbf{F_1}')] = (\frac{3}{52})(\frac{11}{51}) + (\frac{10}{52})(\frac{12}{51}) = \frac{3}{52}$

➤(b).

$\mathbf{D_1}$: the event that a diamond is drawn on the first draw.

$\mathbf{F_2}$: the event that a face card is drawn on the second draw.

E: the event that a diamond is drawn on the first or a face card is selected on second drawing.

$\mathbf{E} = \mathbf{D_1}\cup\mathbf{F_2}$

$P(\mathbf{F_2}) = 12/52$ (See **Problem 3.2**).

$P(\mathbf{E}) = P(\mathbf{D_1}\cup\mathbf{F_2}) = P(\mathbf{D_1}) + P(\mathbf{F_2}) - P(\mathbf{D_1}\cap\mathbf{F_2}) = \frac{13}{52} + \frac{12}{52} - \frac{3}{52} = \frac{22}{52}$

12.3-Problem 4:

➤ **(a)**.

Step 1: $\mathbf{F_1}\cap\mathbf{A_3} = \mathbf{F_1}\cap(\mathbf{A_2}\cup\mathbf{A_2}')\cap\mathbf{A_3} = (\mathbf{F_1}\cap\mathbf{A_2}\cap\mathbf{A_3})\cup(\mathbf{F_1}\cap\mathbf{A_2}'\cap\mathbf{A_3})$, (distributive law)

Step 2: $P(\mathbf{F_1}\cap\mathbf{A_3}) = P(\mathbf{F_1}\cap\mathbf{A_2}\cap\mathbf{A_3}) + P(\mathbf{F_1}\cap\mathbf{A_2}'\cap\mathbf{A_3}) = (12/52)(4/51)(3/50) + (12/52)(47/51)(4/50) = 2400/132600$

Step 3: $P(\mathbf{F_1}\cap\mathbf{A_2}) = (12/52)(4/51) = (12/52)(4/51)(50/50) = 2400/132600$

➤ **(b)**.

Step 1: $\mathbf{E} = \{(\mathbf{F_1}\cap\mathbf{F_2}\cap\mathbf{F_3}')\cup(\mathbf{F_1}\cap\mathbf{F_2}'\cap\mathbf{F_3})\cup(\mathbf{F_1}'\cap\mathbf{F_2}\cap\mathbf{F_3})\}\cup(\mathbf{F_1}\cap\mathbf{F_2}\cap\mathbf{F_3})$, (These sets are disjoint.)

Step 2: $P(\mathbf{E}) = P(\mathbf{F_1}\cap\mathbf{F_2}\cap\mathbf{F_3}') + P(\mathbf{F_1}\cap\mathbf{F_2}'\cap\mathbf{F_3}) + P(\mathbf{F_1}'\cap\mathbf{F_2}\cap\mathbf{F_3}) + P(\mathbf{F_1}\cap\mathbf{F_2}\cap\mathbf{F_3})$

$(12/52)(11/51)(40/50) + (12/52)(40/51)(11/50) + (40/52)(12/51)(11/50) + (12/52)(11/51)(10/50) = 17160/132600$

➤ **(c)**.

Step 1: $\mathbf{E} = (\mathbf{F_1}\cup\mathbf{A_1})\cup(\mathbf{F_2}\cup\mathbf{A_2})\cup(\mathbf{F_3}\cup\mathbf{A_3})$

Step 2: $\mathbf{E}' = [(\mathbf{F_1}\cup\mathbf{A_1})\cup(\mathbf{F_2}\cup\mathbf{A_2})\cup(\mathbf{F_3}\cup\mathbf{A_3})]' = (\mathbf{F_1}'\cap\mathbf{A_1}')\cap(\mathbf{F_2}'\cap\mathbf{A_2}')\cap(\mathbf{F_3}'\cap\mathbf{A_3}')$ (DeMorgan law)

Step 3: $\mathbf{E}' = P[(\mathbf{F_1}'\cap\mathbf{A_1}')\cap(\mathbf{F_2}'\cap\mathbf{A_2}')\cap(\mathbf{F_3}'\cap\mathbf{A_3}')] = (36/52)(35/51)(34/50) = 4284/132600$

Step 4: $P(\mathbf{E}) = 1 - P(\mathbf{E}') = 1 - 4284/132600 = 89760/132600$

➤ **(d)**.

E: The event that at least 1 face card and 1 ace is drawn

$\mathbf{E} = (\mathbf{F}_1 \cup \mathbf{F}_2 \cup \mathbf{F}_3) \cap (\mathbf{A}_1 \cup \mathbf{A}_2 \cup \mathbf{A}_3)$

$\mathbf{E}' = \{(\mathbf{F}_1 \cup \mathbf{F}_2 \cup \mathbf{F}_3) \cap (\mathbf{A}_1 \cup \mathbf{A}_2 \cup \mathbf{A}_3)\}' = (\mathbf{F}_1' \cap \mathbf{F}_2' \cap \mathbf{F}_3') \cup (\mathbf{A}_1' \cap \mathbf{A}_2' \cap \mathbf{A}_3')$, (DeMorgan's law)

$P(\mathbf{E}') = P(\mathbf{F}_1' \cap \mathbf{F}_2' \cap \mathbf{F}_3') + P(\mathbf{A}_1' \cap \mathbf{A}_2' \cap \mathbf{A}_3') - P\{(\mathbf{F}_1' \cap \mathbf{A}_1') \cap (\mathbf{F}_2' \cap \mathbf{A}_2') \cap (\mathbf{F}_3' \cap \mathbf{A}_3')\} =$

$(40/52)(39/51)(38/50) + (48/52)(47/51)(46/50) - (36/52)(35/51)(34/50) = 120216/132600$

$P(\mathbf{E}) = 1 - P(\mathbf{E}) = 1 - 128008/132600 = 4592/132600$

12.3 - Problem 5:
Step 1:
$\mathbf{R}_1$: The event that a red marble was selected from urn 1.
$\mathbf{R}_1'$: The event that a white marble was selected from urn 1.
$\mathbf{W}_2$: The event that a white marble is selected from urn 2.

Step 2: The event that a white marble is selected from urn 2 is equal to the event that a white marble is selected from urn 2 **and** a red marble was selected from urn 1

or

a white marble is selected from urn 2 **and** a white marble was selected from urn 1:

$\mathbf{W}_2 = (\mathbf{R}_1 \cap \mathbf{W}_2) \cup (\mathbf{R}_1' \cap \mathbf{W}_2)$

Step 2: $P(\mathbf{R}_1 \cap \mathbf{W}_2) = P(\mathbf{W}_2 | \mathbf{R}_1) P(\mathbf{R}_1)$

$P(\mathbf{R}_1' \cap \mathbf{W}_2) = P(\mathbf{W}_2 | \mathbf{R}_1') P(\mathbf{R}_1')$.

Step3: $P(\mathbf{W}_2) = P[(\mathbf{R}_1 \cap \mathbf{W}_2) \cup (\mathbf{R}_1' \cap \mathbf{W}_2)] = P(\mathbf{R}_1 \cap \mathbf{W}_2) + P(\mathbf{R}_1' \cap \mathbf{W}_2) = P(\mathbf{W}_2 | \mathbf{R}_1) P(\mathbf{R}_1) + P(\mathbf{W}_2 | \mathbf{R}_1') P(\mathbf{R}_1')$

Step 4: If a red marble was selected from urn 1 and placed in urn 2, then urn 2 will have 4 blue marbles, 10 white marbles and 1 red marble. Therefore,

$P(\mathbf{W}_2 | \mathbf{R}_1) = 4/15$.

Since there are 3 red and 7 white marbles in urn 1, $P(\mathbf{R}_1) = 3/10$.

If a white marble was selected from urn 1 and placed in urn 2, then urn 2 will have 4 blue marbles, and 11 white marbles. Therefore,

$P(\mathbf{W}_2 | \mathbf{R}_1') = 11/15$

$P(\mathbf{R_1}') = 7/10$

Step 5: $P(\mathbf{W_2}) = P(\mathbf{W_2}|\mathbf{R_1})\mathbf{P(R_1)} + P(\mathbf{W_2}|\mathbf{R_1}')\mathbf{P(R_1}') = (\frac{10}{15})(\frac{3}{10}) + (\frac{11}{15})(\frac{7}{10}) = \frac{107}{150}$

12.3 - Problem 6:
Step 1:
S: The event that the teddy bear is profitable.

I: The event that the competing toy company will introduce a talking bear.

Step 2:$P(\mathbf{I})$ = the probability the competing company will introduce the toy =0.60.

$P(\mathbf{S}|\mathbf{I})$ = chance their teddy bear will make a profit for the company provided the competing toy company introduces a talking teddy bear on the market = 0.35.

$P(\mathbf{S}|\mathbf{I}')$ = chance their teddy bear will make a profit for the company provided the competing toy company does not introduce a talking teddy bear on the market = 0.75.

Step 3: $\mathbf{S} = (\mathbf{S}\cap\mathbf{I})\cup(\mathbf{S}\cap\mathbf{I}')$

$P(\mathbf{S}) = P(\mathbf{S}\cap\mathbf{I})+P(\mathbf{S}\cap\mathbf{I}') = P(\mathbf{I})P(\mathbf{S}|\mathbf{I}) + P(\mathbf{I}')P(\mathbf{S}|\mathbf{I}') =(0.60)(0.35)+ (0.40)(0.75) =$

$0.21 + 0.30 = 0.51$.

12.4 - Mutually Independent Events

12.4 - Problem 1:

➤(a).
$P(\mathbf{D}\cap\mathbf{K}') = 12/52$

$P(\mathbf{D}) = 13/52 = 1/4$

$P(\mathbf{K}') = 48/52 = 12/13$

$P\{\mathbf{D}\cap\mathbf{K}'\} = 12/52 = P(\mathbf{D})P(\mathbf{K}')$

➤(b).
$P(\mathbf{K}'|\mathbf{Q}) = 1 \neq P(\mathbf{K}') = 48/52$

12.4 - Problem 2:

➤(a).
$\mathbf{F_1}$**:** the event a face card is drawn on the first drawing.
$\mathbf{D_2}'$**:** the event a diamond is not drawn on the second drawing.

$P(\mathbf{F}_1) = 12/52 = 3/13$

$P(\mathbf{D}_2') = 39/52 = 1/13$ (See **3.3 Example 3**.)

$\mathbf{F}_1 = (\mathbf{F}_1 \cap \mathbf{D}_1') \cup (\mathbf{F}_1 \cap \mathbf{D}_1)$

$P((\mathbf{F}_1 \cap \mathbf{D}_2') = P\{[((\mathbf{F}_1 \cap \mathbf{D}_1') \cup (\mathbf{F}_1 \cap \mathbf{D}_1)] \cap \mathbf{D}_2']\} = P\{[(\mathbf{F}_1 \cap \mathbf{D}_1') \cap \mathbf{D}_2') \cup (\mathbf{F}_1 \cap \mathbf{D}_1) \cap \mathbf{D}_2']\} =$

$P\{[(\mathbf{F}_1 \cap \mathbf{D}_1') \cap \mathbf{D}_2')] + P(\mathbf{F}_1 \cap \mathbf{D}_1) \cap \mathbf{D}_2'] = P(\mathbf{F}_1 \cap \mathbf{D}_1') P\{\mathbf{D}_2' | \mathbf{F}_1 \cap \mathbf{D}_1'\} + P(\mathbf{F}_1 \cap \mathbf{D}_1) P\{\mathbf{D}_2' | \mathbf{F}_1 \cap \mathbf{D}_1\} =$

$(9/52)(38/51) + (3/52)(39/51) = 459/2652 = 153/884 = 9/52$

$P(\mathbf{F}_1) = 12/52$

$P(\mathbf{D}_2') = 39/52$

$P(\mathbf{F}_1)\, P(\mathbf{D}_2') = 9/52 = P((\mathbf{F}_1 \cap \mathbf{D}_2')$

➤(b).
$\mathbf{K}_1$: the event a king is drawn on the first drawing.
$\mathbf{Q}_2'$: the event a queen is not drawn on the second drawing.

$P(\mathbf{Q}_2' | \mathbf{K}_1) = 47/51 \neq P(\mathbf{Q}_2') = 48/52$

12.4 - Problem 3:

D: The event that he or she is a Democrat
Y: The event that this person selected voted for the amendment.

$P(\mathbf{Y}|\mathbf{D}) = 153/255 = 51/85 = 0.60$

$P(\mathbf{Y}) = 322/432 \approx 0.75$

Since $P(\mathbf{Y}|\mathbf{D})$ is not equal to $P(\mathbf{Y})$, the two events are dependent.

12.4 - Problem 4:
Assume **S** is the sample space in order of birth.

Step 1: $\mathbf{S} = \{(g,g),(g,b),(b,g),(b,b)\}$

Step 2: $\mathbf{B} = \{(g,g),(b,b)\}$

Step 3: $\mathbf{G} = \{(g,g)\}$

Step 4: $\mathbf{B} \cap \mathbf{G} = \{(g,g)\} = \mathbf{G}$

Step 5: $\#\mathbf{S} = 4$

#**B** = 2

P(**B**) = 2/4 = 1/2

#**G** = 1

P(**G**) = 1/4

#(B∩G) = #**G** = 1

P(**B∩G**) = 1/4

Step 6: P(**B**)P(**G**) = (1/2)(1/4) = 1/8

P(**B∩G**) ≠ P(**B**)P(**G**).

Therefore, these two events are not independent.

12.4 - Problem 5:

$\mathbf{W_1}$: The event he wins the first game.

$\mathbf{W_2}$: The event he wins the second game.

$\mathbf{W_3}$: The event he wins the third game.

W: The event he wins only two games.

Step 1: He wins game 1, game 2 and losses game 3:$(\mathbf{W_1}\cap\mathbf{W_2}\cap\mathbf{W_3}')$.

or

He wins game 2, game 3 and losses game 1:$(\mathbf{W_1}'\cap\mathbf{W_2}\cap\mathbf{W_3})$.

Step 2: $\mathbf{W} = (\mathbf{W_1}\cap\mathbf{W_2}\cap\mathbf{W_3}')\cup(\mathbf{W_1}'\cap\mathbf{W_2}\cap\mathbf{W_3})$

Step 3: Since it is reasonable to assume the winning or losing of each game are independent of each other,

$P(\mathbf{W_1}\cap\mathbf{W_2}\cap\mathbf{W_3}') = P(\mathbf{W_1})P(\mathbf{W_2})P(\mathbf{W_3}') = (.6)(0.6)(0.4) = 0.144$

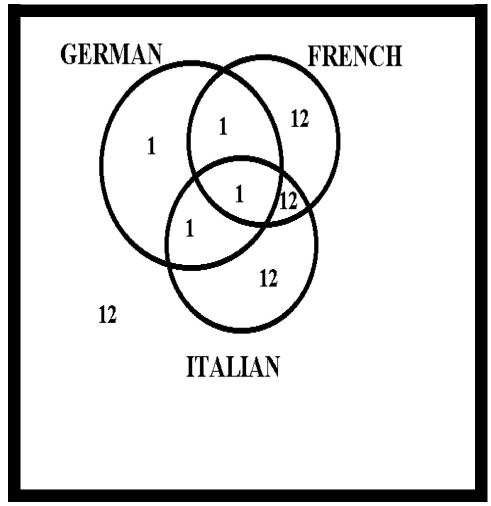
$P(\mathbf{W_1}'\cap\mathbf{W_2}\cap\mathbf{W_3}) = P(\mathbf{W_1}')P(\mathbf{W_2})P(\mathbf{W_3}) = (.4)(0.6)(0.6) = 0.144$.

Step 3:

$\mathbf{P(W)} = P(\mathbf{W_1}\cap\mathbf{W_2}\cap\mathbf{W_3}') + (\mathbf{W_1}'\cap\mathbf{W_2}\cap\mathbf{W_3}) = 0.144 + 0.144 = 0.288$.

12.4 - Problem 6:

From the Venn diagram we find:



#**G** = 4
#**F** = 26
#**I** = 26
#**S** = 52, the cardinality of the sample space **S**.

P(**G**) = 4/52 = 1/13

P(**F**′) = (52 - 26)/52 = 1/2

P(**I**′) = (52 - 26)/52 = 1/2

#(**G** ∩**F**′) = 2

P(**G** ∩**F**′) = 2/52 = 1/26 = (1/13) (1/2) = P(**G**)P(**F**′)

#(**G** ∩ **I**′) = 2

P(**G** ∩**I**′) = 2/52 = 1/26 = (1/13) (1/2) = P(**G**)P(**I**′)

#(**I**′∩**F**′) = #(**I** ∪**F**)′ = 1 + 12 = 13

P(**I**′∩**F**′) = 13/52 = 1/4 = (1/2) (1/2) = P(**I**′)P(**F**′)

Supplementary Problems

1.
Step 1:
$\mathbf{K_1}$: The event that a king is drawn on the first drawing.
$\mathbf{Q_1}$: The event that a queen is drawn on the first drawing.
$\mathbf{K_2}$ The event that a king is drawn on the second drawing.
$\mathbf{Q_2}$: The event that a queen is drawn on the second drawing.

E: The event that a king and queen are drawn:

The first card drawn is a king and the second card drawn is a queen:

$\mathbf{K_1 \cap Q_2}$

or

The first card drawn is a queen and the second card drawn is a king:

$\mathbf{Q_1 \cap K_2}$

Step 2: $\mathbf{E} = (\mathbf{K_1 \cap Q_2}) \cup (\mathbf{Q_1 \cap K_2})$

$P(\mathbf{K_1 \cap Q_2}) = P(\mathbf{K_1})P(\mathbf{Q_2} | \mathbf{K_1}) = (4/52)(4/51) = \dfrac{16}{2652}$

$P(\mathbf{Q_1 \cap K_2}) = P(\mathbf{Q_1})P(\mathbf{K_2} | \mathbf{Q_1}) = (4/52)(4/51) = \dfrac{16}{2652}$

$P(\mathbf{E}) = P[(\mathbf{K_1 \cap Q_2}) \cup (\mathbf{Q_1 \cap K_2})] = P(\mathbf{K_1 \cap Q_2}) + P(\mathbf{Q_1 \cap K_2}) = \dfrac{32}{2652} = \dfrac{8}{663}$.

2.

$\mathbf{K_1}$: The event that a king is drawn on the first drawing.

$\mathbf{D_1}$: The event that a diamond is drawn on the first drawing.

$\mathbf{D_2}$: The event that a diamond is drawn on the second drawing.

$\mathbf{E} = \mathbf{K_1 \cap D_2}$ = the event that the first card is a king and the second card is a diamond.

Step 2: $\mathbf{K_1} = (\mathbf{K_1 \cap D_1}) \cup (\mathbf{K_1 \cap D_1}')$

Step 3: $\mathbf{E} = \mathbf{K_1 \cap D_2} = [(\mathbf{K_1 \cap D_1}) \cup (\mathbf{K_1 \cap D_1}')] \cap \mathbf{D_2} = [(\mathbf{K_1 \cap D_1}) \cap \mathbf{D_2}] \cup [(\mathbf{K_1 \cap D_1}') \cap \mathbf{D_2}]$

Step 4: $P(\mathbf{E})$ =

$P[(\mathbf{K_1 \cap D_1}) \cap \mathbf{D_2}] + P[(\mathbf{K_1 \cap D_1}') \cap \mathbf{D_2}] = P(\mathbf{K_1 \cap D_1})P[\mathbf{D_2} | (\mathbf{K_1 \cap D_1})] + P(\mathbf{K_1 \cap D_1}')P[\mathbf{D_2} | (\mathbf{K_1 \cap D_1}')]$

$$= \left(\frac{1}{52}\right)\left(\frac{12}{51}\right) + \left(\frac{3}{52}\right)\left(\frac{13}{51}\right) = \frac{1}{52}$$

3.

$\mathbf{R}_i$ (i = 1,2,3): The event that the ith drawing is a red marble.

$\mathbf{W}_i$ (i = 1,2,3): The event that the ith drawing is a white marble.

$\mathbf{B}_i$ (i = 1,2,3): The event that the ith drawing is a blue marble.

$\mathbf{E}$: the event that all three marbles selected are the same color.

Step 1: $\mathbf{R_1 \cap R_2 \cap R_3}$: All three drawings are red

or

$W_1 \cap \mathbf{W_2 \cap W_3}$: All three drawings are white

or

$\mathbf{B_1 \cap B_2 \cap B_3}$: All three drawings are blue.

Step 2: $\mathbf{E = (R_1 \cap R_2 \cap R_3)} \cup (W_1 \cap \mathbf{W_2 \cap W_3}) \cup (\mathbf{B_1 \cap B_2 \cap B_3})$

Step 3: $P(\mathbf{R_1 \cap R_2 \cap R_3}) = P(\mathbf{R_1})P(\mathbf{R_2}|\mathbf{R_1})P(\mathbf{R_3}|\mathbf{R_1 \cap R_2}) = (\frac{25}{40})(\frac{24}{39})(\frac{23}{38}) = \frac{13800}{59280}$

$P(\mathbf{W_1 \cap W_2 \cap W_3}) = P(\mathbf{W_1})P(\mathbf{W_2}|\mathbf{W_1})P(\mathbf{W_3}|\mathbf{W_1 \cap W_2}) = (\frac{5}{40})(\frac{4}{39})(\frac{3}{38}) = \frac{60}{59280}$

$P(\mathbf{B_1 \cap B_2 \cap B_3}) = P(\mathbf{B_1})P(\mathbf{B_2}|\mathbf{B_1})P(\mathbf{B_3}|\mathbf{B_1 \cap B_2}) = (\frac{10}{40})(\frac{9}{39})(\frac{8}{38}) = \frac{720}{59280}$

Step 4: $P(\mathbf{E}) = P(\mathbf{R_1 \cap R_2 \cap R_3}) + P(W_1 \cap \mathbf{W_2 \cap W_3}) + P(\mathbf{B_1 \cap B_2 \cap B_3}) =$

$$(\frac{25}{40})(\frac{24}{39})(\frac{23}{38}) = \frac{13800 + 60 + 720}{59280} = \frac{243}{988}$$

4.
Step 1:
$\mathbf{K_1}$: The event that a king is drawn on the first drawing.
$\mathbf{Q_1}$: The event that a queen is drawn on the first drawing.
$\mathbf{K_2}$ The event that a king is drawn on the second drawing.
$\mathbf{Q_2}$: The event that a queen is drawn on the second drawing.

E = the event that the first card is a king or queen and the second card is a king or queen =

$(\mathbf{K_1 \cup Q_1}) \cap (\mathbf{K_2 \cup Q_2})$.

$\mathbf{E} = (\mathbf{K_1 \cup Q_1}) \cap (\mathbf{K_2 \cup Q_2}) = (\mathbf{K_1 \cap K_2}) \cup (\mathbf{K_1 \cap Q_2}) \cup (\mathbf{Q_1 \cap K_2}) \cup (\mathbf{Q_1 \cap Q_2})$

$P(\mathbf{E}) = P(\mathbf{K_1 \cap K_2}) + P(\mathbf{K_1 \cap Q_2}) + P(\mathbf{Q_1 \cap K_2}) + P(\mathbf{Q_1 \cap Q_2}) =$

$P(\mathbf{K_1})P(\mathbf{K_2}|\mathbf{K_1}) + P(\mathbf{K_1})P(\mathbf{Q_2}|\mathbf{K_1}) + P(\mathbf{Q_1})P(\mathbf{K_2}|\mathbf{Q_1}) + P(\mathbf{Q_1})P(\mathbf{Q_2}|\mathbf{Q_1}) =$

$$(\frac{4}{52})(\frac{3}{51}) + (\frac{4}{52})(\frac{4}{51}) + (\frac{4}{52})(\frac{4}{51}) + (\frac{4}{52})(\frac{3}{51}) = \frac{14}{663}$$

5.
$\mathbf{K_1}$: The event that a king is drawn on the first drawing.
$\mathbf{Q_1}$: The event that a queen is drawn on the first drawing.

$\mathbf{K_2}$ The event that a king is drawn on the second drawing.
$\mathbf{Q_2}$: The event that a queen is drawn on the second drawing.

$\mathbf{E} = (\mathbf{K_1} \cup \mathbf{Q_1}) \cup (\mathbf{K_2} \cup \mathbf{Q_2})$

$\mathbf{E}' = (\mathbf{K_1}' \cap \mathbf{Q_1}') \cap (\mathbf{K_2}' \cap \mathbf{Q_2}')$

$P(\mathbf{E}') = P[(\mathbf{K_1}' \cap \mathbf{Q_1}') \cap (\mathbf{K_2}' \cap \mathbf{Q_2}')] = P(\mathbf{K_1}' \cap \mathbf{Q_1}')P(\mathbf{K_2}' \cap \mathbf{Q_2}' | \mathbf{K_1}' \cap \mathbf{Q_1}') = (\frac{44}{52})(\frac{43}{51}) = \frac{473}{663}$

$P(\mathbf{E}) = 1 - \frac{473}{663} = \frac{190}{663}$

6.
➤ a.
$\mathbf{D_1}$: The event that the first chip tested is defective.
$\mathbf{D_2}$: The event that the second chip tested is defective.
$\mathbf{D_3}$: The event that the third chip tested is defective.
$\mathbf{E}$: The event that he will not return the box.

From the statement of the problem, he will only return the box if at least one chip that he tested is defective. Therefore, **E** equals the event that none of the $(\frac{95}{100})(\frac{94}{99})(\frac{93}{98}) = \frac{27683}{32340}$ chips tested will be found defective:

$\mathbf{E} = \mathbf{D_1}' \cap \mathbf{D_2}' \cap \mathbf{D_3}'$

$P(\mathbf{E}) = P(\mathbf{D_1}')P(\mathbf{D_2}' | \mathbf{D_1}')P(\mathbf{D_3}' | \mathbf{D_1}' \cap \mathbf{D_2}')$

➤ b.
E: the event that at least two of the three chips selected are defective.

Exactly two are defective:

$(\mathbf{D_1}' \cap \mathbf{D_2} \cap \mathbf{D_3}) \cup (\mathbf{D_1} \cap \mathbf{D_2}' \cap \mathbf{D_3}) \cup (\mathbf{D_1} \cap \mathbf{D_2} \cap \mathbf{D_3}')$

or

all three are defective:

$\mathbf{D_1} \cap \mathbf{D_2} \cap \mathbf{D_3}$
$\mathbf{E} = (\mathbf{D_1}' \cap \mathbf{D_2} \cap \mathbf{D_3}) \cup (\mathbf{D_1} \cap \mathbf{D_2}' \cap \mathbf{D_3}) \cup (\mathbf{D_1} \cap \mathbf{D_2} \cap \mathbf{D_3}') \cup (\mathbf{D_1} \cap \mathbf{D_2} \cap \mathbf{D_3})$

$P(\mathbf{E}) = P[(\mathbf{D_1}' \cap \mathbf{D_2} \cap \mathbf{D_3}) \cup (\mathbf{D_1} \cap \mathbf{D_2}' \cap \mathbf{D_3}) \cup (\mathbf{D_1} \cap \mathbf{D_2} \cap \mathbf{D_3}') \cup (\mathbf{D_1} \cap \mathbf{D_2} \cap \mathbf{D_3})] =$

$P(\mathbf{D_1}'\cap\mathbf{D_2}\cap\mathbf{D_3}) + P(\mathbf{D_1}\cap\mathbf{D_2}'\cap\mathbf{D_3}) + P(\mathbf{D_1}\cap\mathbf{D_2}\cap\mathbf{D_3}') + P(\mathbf{D_1}\cap\mathbf{D_2}\cap\mathbf{D_3}) =$

$$(\frac{95}{100})(\frac{5}{99})(\frac{4}{98}) + (\frac{5}{100})(\frac{95}{99})(\frac{4}{98}) + (\frac{5}{100})(\frac{4}{99})(\frac{95}{98}) + (\frac{5}{100})(\frac{4}{99})(\frac{3}{98}) = \frac{16}{2695}$$

7.
$\mathbf{U_1}$: The event urn 1 is selected.
$\mathbf{U_2}$: The event urn 2 is selected.
$\mathbf{U_3}$: The event urn 3 is selected.
W: The event a white marble is selected.

This event can happen in the following ways:

A white marble is selected and it came from urn 1:

$\mathbf{W}\cap\mathbf{U_1}$

or

a white marble is selected and it came from urn 2:

$\mathbf{W}\cap\mathbf{U_2}$

or

a white marble is selected and it came from urn 3:

$\mathbf{W}\cap\mathbf{U_3}$

Therefore, $\mathbf{W} = (\mathbf{W}\cap\mathbf{U_1})\cup(\mathbf{W}\cap\mathbf{U_2})\cup(\mathbf{W}\cap\mathbf{U_3})$

$P(\mathbf{E}) = P[(\mathbf{W}\cap\mathbf{U_1})\cup(\mathbf{W}\cap\mathbf{U_2})\cup(\mathbf{W}\cap\mathbf{U_3})] = P(\mathbf{W}\cap\mathbf{U_1}) + P(\mathbf{W}\cap\mathbf{U_2}) + P(\mathbf{W}\cap\mathbf{U_3}) =$

$P(\mathbf{U_1})P(\mathbf{W}|\mathbf{U_1}) + P(\mathbf{U_2})P(\mathbf{W}|\mathbf{U_2}) + P(\mathbf{U_2})P(\mathbf{W}|\mathbf{U_3}) =$

$$(\frac{1}{6})(\frac{7}{10}) + (\frac{2}{6})(\frac{10}{15}) + (\frac{3}{6})(\frac{10}{20}) = \frac{53}{90}$$

8.
$\mathbf{K_1}$: The event that the first card drawn is a king.
$\mathbf{D_1}$: The event that the first card drawn is a diamond.
$\mathbf{K_2}$: The event that the second card drawn is a king.

E: The event that the first card drawn is a king or diamond and the second card is a king can occur as follows: king of diamonds is drawn on the first drawing **and** a king is drawn on the second drawing:

$(\mathbf{K_1}\cap\mathbf{D_1})\cap\mathbf{K_2}$

or

a king that is not a diamond is drawn on the first drawing **and** a king is drawn on the second drawing:

$\mathbf{K_1}\cap\mathbf{D_1}')\cap\mathbf{K_2}$

or

a diamond that is not a king is drawn on the first drawing **and** a king is drawn on the second drawing:

$(\mathbf{D_1}\cap\mathbf{K_1}')\cap\mathbf{K_2}$

Therefore,

$$\mathbf{E} = [(\mathbf{K_1}\cap\mathbf{D_1})\cap\mathbf{K_2}]\cup[(\mathbf{K_1}\cap\mathbf{D_1}')\cap\mathbf{K_2}]\cup[(\mathbf{D_1}\cap\mathbf{K_1}')]\cap\mathbf{K_2}]$$

$$P(\mathbf{E}) = P([(\mathbf{K_1}\cap\mathbf{D_1})\cap\mathbf{K_2}]\cup[(\mathbf{K_1}\cap\mathbf{D_1}')\cap\mathbf{K_2}]\cup[(\mathbf{D_1}\cap\mathbf{K_1}')]\cap\mathbf{K_2}]) =$$

$$P[(\mathbf{K_1}\cap\mathbf{D_1})\cap\mathbf{K_2}] + P[(\mathbf{K_1}\cap\mathbf{D_1}')\cap\mathbf{K_2}] + P[(\mathbf{D_1}\cap\mathbf{K_1}')\cap\mathbf{K_2}]) =$$

$$P(\mathbf{K_1}\cap\mathbf{D_1})P(\mathbf{K_2}|\mathbf{K_1}\cap\mathbf{D_1}) + P(\mathbf{K_1}\cap\mathbf{D_1}')P(\mathbf{K_2}|\mathbf{K_1}\cap\mathbf{D_1}') + P(\mathbf{D_1}\cap\mathbf{K_1}')P(\mathbf{K_2}|\mathbf{D_1}\cap\mathbf{K_1}') =$$

$$\left(\frac{1}{52}\right)\left(\frac{3}{51}\right) + \left(\frac{3}{52}\right)\left(\frac{3}{51}\right) + \left(\frac{12}{52}\right)\left(\frac{4}{51}\right) = \frac{5}{221}$$

9.

➤ a.

$\mathbf{W_1}$: The event that he won the first game.
$\mathbf{W_2}$: The event that he won the second game.
$\mathbf{W_3}$: The event that he won the third game.
E: The event he will stop betting by the third game:

He wins the first game:

$\mathbf{W_1}$

or

He losses the first and wins the second game:

$\mathbf{W_1' \cap W_2}$

or

He losses the first, second games but wins the third game:

$W_1' \cap \mathbf{W_2' \cap W_3}$

$\mathbf{E} = \mathbf{W_1} \cup (\mathbf{W_1' \cap W_2}) \cup (\mathbf{W_1' \cap W_2' \cap W_3})$

$P(\mathbf{E}) = P(\mathbf{W_1}) + P(\mathbf{W_1' \cap W_2}) + P(\mathbf{W_1' \cap W_2' \cap W_3}) = P(\mathbf{W_1}) + P(\mathbf{W_1'})P(\mathbf{W_2}) + P(\mathbf{W_1'})P(\mathbf{W_2'})P(\mathbf{W_3}) =$

$0.6 + (0.40)(0.60) + (0.40)(0.40)(0.60) = 0.936$

➤ b.
E: The event that he will stop betting on the third game. This can only happen if he losses the first two games and wins the third game:

$\mathbf{E} = \mathbf{W_1' \cap W_2' \cap W_3}$

$P(\mathbf{E}) = P(\mathbf{W_1' \cap W_2' \cap W_3}) = (0.4)(0.40)(0.60) = 0.096.$

10.
$\mathbf{B_1}$: The event that the marble selected from urn 1 is blue.
$\mathbf{B_2}$: The event that the marble selected from urn 2 is blue.
$\mathbf{B_3}$: The event that the marble selected from urn 3 is blue.

The event that the marble selected from urn 3 is blue can occur in the following ways:

A blue marble is selected from urn 1, urn 2 and urn 3:

$\mathbf{B_1 \cap B_2 \cap B_3}$

or

A blue marble is selected from urn 1 and urn 3 but not from urn 2:

$\mathbf{B_1 \cap B_2' \cap B_3}$

or

A blue marble is selected from urn 2 and urn 3 but not from urn 1:

$\mathbf{B_1' \cap B_2 \cap B_3}$

or

A blue marble is selected from urn 3 but not from urn 1 and urn 2:

$\mathbf{B_1' \cap B_2' \cap B_3}$

$\mathbf{B_3} = (\mathbf{B_1 \cap B_2 \cap B_3}) \cup (\mathbf{B_1 \cap B_2' \cap B_3}) \cup (\mathbf{B_1' \cap B_2 \cap B_3}) \cup (\mathbf{B_1' \cap B_2' \cap B_3})$

$P(\mathbf{B_3}) = P[(\mathbf{B_1 \cap B_2 \cap B_3}) \cup (\mathbf{B_1 \cap B_2' \cap B_3}) \cup (\mathbf{B_1' \cap B_2 \cap B_3}) \cup (\mathbf{B_1' \cap B_2' \cap B_3})] =$

$P[(\mathbf{B_1 \cap B_2 \cap B_3}) + P(\mathbf{B_1 \cap B_2' \cap B_3}) + P(\mathbf{B_1' \cap B_2 \cap B_3}) + P(\mathbf{B_1' \cap B_2' \cap B_3})] =$

$P(\mathbf{B_1})P(\mathbf{B_2}|\mathbf{B_1})P(\mathbf{B_3}|\mathbf{B_1 \cap B_2}) + P(\mathbf{B_1})P(\mathbf{B_2'}|\mathbf{B_1})P(\mathbf{B_3}|\mathbf{B_1 \cap B_2'}) + P(\mathbf{B_1'})P(\mathbf{B_2}|\mathbf{B_1'})P(\mathbf{B_3}|\mathbf{B_1' \cap B_2}) +$

$P(\mathbf{B_1'})P(\mathbf{B_2'}|\mathbf{B_1'})P(\mathbf{B_3}|\mathbf{B_1' \cap B_2'}) =$

$$(\frac{3}{10})(\frac{5}{15})(\frac{6}{11}) + (\frac{3}{10})(\frac{10}{15})(\frac{5}{11}) + (\frac{7}{10})(\frac{4}{15})(\frac{6}{11}) + (\frac{7}{10})(\frac{11}{15})(\frac{5}{11}) = \frac{793}{2443} \; .$$

11.
$\mathbf{D_1}$: The event a diamond is selected from the first deck and placed in the second deck.
$\mathbf{D_2}$: The event a diamond is selected from the second deck and placed in the first deck.
$\mathbf{D_3}$: The event a diamond is selected on the final drawing.

A diamond is selected on the first, second and final drawing:

$\mathbf{D_1 \cap D_2 \cap D_3}$

or

A diamond is selected on the first and third drawing but not on the second drawing:

$\mathbf{D_1 \cap D_2' \cap D_3}$

or

A diamond is selected on the second and third drawing but not on the first drawing:

$\mathbf{D_1'\cap D_2\cap D_3}$

or

A diamond is selected only on the third drawing:

$\mathbf{D_1'\cap D_2'\cap D_3}$

$\mathbf{D}_3 = (\mathbf{D_1\cap D_2\cap D_3})\cup(\mathbf{D_1\cap D_2'\cap D_3})\cup(\mathbf{D_1'\cap D_2\cap D_3})\cup(\mathbf{D_1'\cap D_2'\cap D_3})$

$P(\mathbf{D_3}) = P[(\mathbf{D_1\cap D_2\cap D_3})\cup(\mathbf{D_1\cap D_2'\cap D_3})\cup(\mathbf{D_1'\cap D_2\cap D_3})\cup(\mathbf{D_1'\cap D_2'\cap D_3})] =$

$P(\mathbf{D_1\cap D_2\cap D_3}) + P(\mathbf{D_1\cap D_2'\cap D_3}) + P(\mathbf{D_1'\cap D_2\cap D_3}) + P(\mathbf{D_1'\cap D_2'\cap D_3})] =$

$P(\mathbf{D_1})P(\mathbf{D_2}|\mathbf{D_1})P(\mathbf{D_3}|\mathbf{D_1\cap D_2}) + P(\mathbf{D_1})P(\mathbf{D_2'}|\mathbf{D_1})P(\mathbf{D_3}|\mathbf{D_1\cap D_2'}) +$

$P(\mathbf{D_1'})P(\mathbf{D_2}|\mathbf{D_1'})P(\mathbf{D_3}|\mathbf{D_1'\cap D_2}) + P(\mathbf{D_1'})P(\mathbf{D_2'}|\mathbf{D_1'})P(\mathbf{D_3}|\mathbf{D_1'\cap D_2'}) =$

$$(\frac{13}{52})(\frac{14}{53})(\frac{13}{52}) + (\frac{13}{52})(\frac{39}{53})(\frac{12}{52}) + (\frac{39}{52})(\frac{13}{53})(\frac{14}{52}) + (\frac{39}{52})(\frac{40}{53})(\frac{13}{52}) = \frac{1}{4}$$

12.
$\mathbf{R_1}$: The event a red marble is selected from urn 1 and placed in urn 2.
$\mathbf{R_2}$: The event a red marble is selected from urn 2 and placed in urn 1.
T: The event a 3 occurs on tossing the die.
B: The event a black marble is eventually selected:

A red marble is selected from urn 1 and placed in urn 2 and a black marble is finally selected:

$\mathbf{T\cap R_1\cap B}$

or

A black marble is selected from urn 1 and placed in urn 2 and a black marble is finally selected:

$\mathbf{T\cap R_1'\cap B}$
or

A red marble is selected from urn 2 and placed in urn 1 and a black marble is finally selected:

$\mathbf{T}'\cap\mathbf{R_1}\cap\mathbf{B}$

or

A black marble is selected from urn 2 and placed in urn 1 and a black marble is finally selected:

$\mathbf{T}'\cap\mathbf{R_1}'\cap\mathbf{B}$

$\mathbf{B} = (\mathbf{T}\cap\mathbf{R_1}\cap\mathbf{B})\cup(\mathbf{T}\cap\mathbf{R_1}'\cap\mathbf{B})\cup(\mathbf{T}'\cap\mathbf{R_1}\cap\mathbf{B})\cup(\mathbf{T}'\cap\mathbf{R_1}'\cap\mathbf{B})$

$P(\mathbf{B}) = P(\mathbf{T}\cap\mathbf{R_1}\cap\mathbf{B}) + P(\mathbf{T}\cap\mathbf{R_1}\cap\mathbf{B}) + P(\mathbf{T}'\cap\mathbf{R_1}\cap\mathbf{B}) + P(\mathbf{T}'\cap\mathbf{R_1}'\cap\mathbf{B}) =$

$P(\mathbf{T})P(\mathbf{R_1}|\mathbf{T})P(\mathbf{B}|\mathbf{T}\cap\mathbf{R_1}) + P(\mathbf{T})P(\mathbf{R_1}'|\mathbf{T})P(\mathbf{B}|\mathbf{T}\cap\mathbf{R_1}') + P(\mathbf{T}')P(\mathbf{R_1}|\mathbf{T}')P(\mathbf{B}|\mathbf{T}'\cap\mathbf{R_1}) +$

$P(\mathbf{T}')P(\mathbf{R_1}'|\mathbf{T}')P(\mathbf{B}|\mathbf{T}'\cap\mathbf{R_1}') =$

$$(\frac{1}{6})(\frac{1}{2})(\frac{2}{10}) + (\frac{1}{6})(\frac{1}{2})(\frac{3}{10}) + (\frac{5}{6})(\frac{7}{9})(\frac{5}{11}) + (\frac{5}{6})(\frac{2}{9})(\frac{6}{11}) = \frac{1039}{2376}$$

13.

$\mathbf{M_1}$: The event the first student selected is a male.
$\mathbf{M_2}$: The event the second student selected is a male.
$\mathbf{M_3}$: The event the third student selected is a male.
$\mathbf{M_4}$: The event the fourth student selected is a male.
$\mathbf{E}$: The event that this process will stop on the fourth selection:

A male is selected only on the first and fourth selection:

$\mathbf{M_1}\cap\mathbf{M_2}'\cap\mathbf{M_3}'\cap\mathbf{M_4}$

or

A male is selected only on the second and fourth selection:

$\mathbf{M_1}'\cap\mathbf{M_2}\cap\mathbf{M_3}'\cap\mathbf{M_4}$

or

A male is selected only on the third and fourth selection:

$\mathbf{M_1}'\cap\mathbf{M_2}'\cap\mathbf{M_3}\cap\mathbf{M_4}$

$\mathbf{E} = (\mathbf{M_1}\cap\mathbf{M_2}'\cap\mathbf{M_3}'\cap\mathbf{M_4})\cup(\mathbf{M_1}'\cap\mathbf{M_2}\cap\mathbf{M_3}'\cap\mathbf{M_4})\cup(\mathbf{M_1}'\cap\mathbf{M_2}'\cap\mathbf{M_3}\cap\mathbf{M_4})$

$P(\mathbf{E}) = P[(\mathbf{M_1}\cap\mathbf{M_2}'\cap\mathbf{M_3}'\cap\mathbf{M_4})\cup(\mathbf{M_1}'\cap\mathbf{M_2}\cap\mathbf{M_3}'\cap\mathbf{M_4})\cup(\mathbf{M_1}'\cap\mathbf{M_2}'\cap\mathbf{M_3}\cap\mathbf{M_4})] =$

$P(\mathbf{M_1}\cap\mathbf{M_2}'\cap\mathbf{M_3}'\cap\mathbf{M_4}) + P(\mathbf{M_1}'\cap\mathbf{M_2}\cap\mathbf{M_3}'\cap\mathbf{M_4}) + P(\mathbf{M_1}'\cap\mathbf{M_2}'\cap\mathbf{M_3}\cap\mathbf{M_4}) =$

$P(\mathbf{M_1})P(\mathbf{M_2}'|\mathbf{M_1})P(\mathbf{M_3}'|\mathbf{M_1}\cap\mathbf{M_2}')P(\mathbf{M_4}|\mathbf{M_1}\cap\mathbf{M_2}'\cap\mathbf{M_3}') +$

$P(\mathbf{M_1}')P(\mathbf{M_2}|\mathbf{M_1}')P(\mathbf{M_3}'|\mathbf{M_1}'\cap\mathbf{M_2})P(\mathbf{M_4}|\mathbf{M_1}'\cap\mathbf{M_2}\cap\mathbf{M_3}') +$

$P(\mathbf{M_1}')P(\mathbf{M_2}'|\mathbf{M_1}')P(\mathbf{M_3}|\mathbf{M_1}'\cap\mathbf{M_2}')P(\mathbf{M_4}|\mathbf{M_1}'\cap\mathbf{M_2}'\cap\mathbf{M_3}) =$

$$(\frac{40}{100})(\frac{60}{99})(\frac{59}{98})(\frac{39}{97}) + (\frac{60}{100})(\frac{40}{99})(\frac{59}{98})(\frac{39}{97}) + (\frac{60}{100})(\frac{59}{99})(\frac{40}{98})(\frac{39}{97}) = \frac{9204}{52283}$$

14.

Step 1: Assume **A** and **B** are non-empty events that are independent and mutually exclusive.

Step 2: Since they are mutually exclusive, $P(\mathbf{A}\cap\mathbf{B}) = P(\phi) = 0$

Step 3: Since **A** and **B** are independent, $P(\mathbf{B}|\mathbf{A}) = \dfrac{P(\mathbf{A}\cap\mathbf{B})}{P(\mathbf{B})} = P(\mathbf{B}) > 0$,

Step 4: Therefore, $P(\mathbf{A}\cap\mathbf{B}) > 0$.

Step 5: The results in Step 2 and Step 4 are in contradictions.

Step 6: Therefore, two non-empty events cannot be both mutually exclusive and independent.

Step 7: Non-empty mutually exclusive events are always dependent.

15.

Step 1: $P(\mathbf{K_3}|\mathbf{K_1}) = \dfrac{P(\mathbf{K_1}\cap\mathbf{K_3})}{P(\mathbf{K_1})}$

Step 2: $\mathbf{K_1}\cap\mathbf{K_3} = (\mathbf{K_1}\cap\mathbf{K_2}\cap\mathbf{K_3})\cup(\mathbf{K_1}\cap\mathbf{K_2}'\cap\mathbf{K_3})$

$P(\mathbf{K_1}\cap\mathbf{K_3}) = P[(\mathbf{K_1}\cap\mathbf{K_2}\cap\mathbf{K_3})\cup(\mathbf{K_1}\cap\mathbf{K_2}'\cap\mathbf{K_3})] = P(\mathbf{K_1}\cap\mathbf{K_2}\cap\mathbf{K_3}) + P(\mathbf{K_1}\cap\mathbf{K_2}'\cap\mathbf{K_3}) =$

$$(\frac{4}{52})(\frac{3}{51})(\frac{2}{50}) + (\frac{4}{52})(\frac{48}{52})(\frac{3}{50}) = \frac{1}{221}$$

$P(\mathbf{K_1}) = 4/52 = 1/13$.

Step 3: $P(\mathbf{K_3}|\mathbf{K_1}) = \dfrac{P(\mathbf{K_1} \cap \mathbf{K_3})}{P(\mathbf{K_1})} = \dfrac{13}{221} = \dfrac{1}{17}.$

Stelp 4: $P(\mathbf{K_1}|\mathbf{K_3}) = \dfrac{P(\mathbf{K_1} \cap \mathbf{K_3})}{P(\mathbf{K_3})} \dfrac{(\frac{1}{221})}{P(\mathbf{K_3})}$

Step 5: $\mathbf{K_3} = (\mathbf{K_1} \cap \mathbf{K_2} \cap \mathbf{K_3}) \cup (\mathbf{K_1}' \cap \mathbf{K_2} \cap \mathbf{K_3}) \cup (\mathbf{K_1} \cap \mathbf{K_2}' \cap \mathbf{K_3}) \cup (\mathbf{K_1}' \cap \mathbf{K_2}' \cap \mathbf{K_3})$

$P(\mathbf{K_3}) = P(\mathbf{K_1} \cap \mathbf{K_2} \cap \mathbf{K_3}) + P(\mathbf{K_1}' \cap \mathbf{K_2} \cap \mathbf{K_3}) + P(\mathbf{K_1} \cap \mathbf{K_2}' \cap \mathbf{K_3}) + P(\mathbf{K_1}' \cap \mathbf{K_2}' \cap \mathbf{K_3}))$

$$(\frac{4}{52})(\frac{3}{51})(\frac{2}{50}) + (\frac{48}{52})(\frac{4}{51})(\frac{3}{50}) + (\frac{4}{52})(\frac{48}{51})(\frac{3}{50}) + (\frac{48}{52})(\frac{47}{51})(\frac{4}{50})) = \frac{1}{13}$$

Step 6: $P(\mathbf{K_1}|\mathbf{K_3}) = \dfrac{P(\mathbf{K_1} \cap \mathbf{K_3})}{P(\mathbf{K_3})} \dfrac{(\frac{1}{221})}{(\frac{1}{13})} = \dfrac{1}{17} = P(\mathbf{K_3}|\mathbf{K_1}).$

Conclusion: Yes, the two conditional probabilities are the same.

16.

➤ a.

Step 1: $1 = P(\mathbf{S}|\mathbf{A}) = P(\mathbf{B} \cup \mathbf{B}'|\mathbf{A}) = \dfrac{P[(\mathbf{B} \cup \mathbf{B}') \cap \mathbf{A}]}{P(\mathbf{A})} = \dfrac{P[(\mathbf{B} \cap A) \cup (\mathbf{B}' \cap \mathbf{A}))]}{P(\mathbf{A})} =$

$\dfrac{P[(\mathbf{B} \cap A) + (\mathbf{B}' \cap \mathbf{A}))]}{P(\mathbf{A})} = \dfrac{P(\mathbf{B} \cap A)}{P(\mathbf{A})} + \dfrac{(\mathbf{B}' \cap \mathbf{A})}{P(\mathbf{A})} = P(\mathbf{B}|\mathbf{A}) + P(\mathbf{B}'|\mathbf{A}).$

Therefore, $P(\mathbf{B}'|\mathbf{A}) = 1 - P(\mathbf{B}|\mathbf{A})$.

Step 2: Since **A** and **B** are independent, $P(\mathbf{B}|\mathbf{A}) = \mathbf{B}$.

From Step 1, we have $P(\mathbf{B}'|\mathbf{A}) = 1 - P(\mathbf{B}|\mathbf{A}) = 1 - P(\mathbf{B}) = P(\mathbf{B}')$.

➤ b.

$P(\mathbf{B}'|\mathbf{A}') = 1 - P(\mathbf{B}|\mathbf{A}')$

Since in a. we showed $\mathbf{A}'$ and **B** are independent, $P(\mathbf{B}'|\mathbf{A}') = 1 - P(\mathbf{B}|\mathbf{A}') = 1 - P(\mathbf{B}) = P(\mathbf{B}')$.

17.
$\mathbf{W}_1$: The first drawing is white.
$\mathbf{W}_2$: The second drawing is white.
$\mathbf{D}_1$: 1 or 2 occurs
$\mathbf{D}_2$: 3,4 or 5 occurs
$\mathbf{D}_3$: 6 occurs

The event $\mathbf{E} = \mathbf{W}_1 \cap \mathbf{W}_2$ that both marbles are white can occur as follows:

Urn A is selected: $\mathbf{D}_1 \cap (\mathbf{W}_1 \cap \mathbf{W}_2)$

or

Urn B is selected:

$\mathbf{D}_2 \cap (\mathbf{W}_1 \cap \mathbf{W}_2)$

or
Both urn A and B are selected: $\mathbf{D}_3 \cap (\mathbf{W}_1 \cap \mathbf{W}_2)$.

$$\mathbf{E} = [\mathbf{D}_1 \cap (\mathbf{W}_1 \cap \mathbf{W}_2)] \cup [\mathbf{D}_2 \cap (\mathbf{W}_1 \cap \mathbf{W}_2)] \cup [\mathbf{D}_3 \cap (\mathbf{W}_1 \cap \mathbf{W}_2)]$$

$$P(\mathbf{E}) = P[\mathbf{D}_1 \cap (\mathbf{W}_1 \cap \mathbf{W}_2)] + P[\mathbf{D}_2 \cap (\mathbf{W}_1 \cap \mathbf{W}_2)] + P[\mathbf{D}_3 \cap (\mathbf{W}_1 \cap \mathbf{W}_2)] =$$

$$P(\mathbf{D}_1)P[(\mathbf{W}_1 \cap \mathbf{W}_2) | \mathbf{D}_1] + P(\mathbf{D}_2)P[(\mathbf{W}_1 \cap \mathbf{W}_2) | \mathbf{D}_2] + P(\mathbf{D}_3)P[(\mathbf{W}_1 \cap \mathbf{W}_2) | \mathbf{D}_3] =$$

$$(\frac{2}{6})(\frac{6}{10})(\frac{5}{9}) + (\frac{3}{6})(\frac{3}{10})(\frac{2}{9}) + (\frac{1}{6})(\frac{6}{10})(\frac{3}{10})\ ) = \frac{157}{900}$$

18. .
Step 1: $P(\mathbf{E} | \mathbf{A} \cup \mathbf{B}) =$

$$\frac{\mathbf{P[E \cap (A \cup B)]}}{\mathbf{P(A \cup B)}} = \frac{\mathbf{P[(E \cap A) \cup (E \cap B)]}}{\mathbf{P(A \cup B)}} = \frac{\mathbf{P[(E \cap A)\ +\ (E \cap B)\ -\ P(E \cap A \cap B)]}}{\mathbf{P(A)\ +\ P(B)\ -\ P(A \cap B))}}$$

19.

$$P(\mathbf{A} \cup \mathbf{B} | \mathbf{E}) = \frac{\mathbf{P(E \cap A)}}{\mathbf{P}(E)} + \frac{\mathbf{P(E \cap B)}}{P(\mathbf{E})} - \frac{\mathbf{P(E \cap A \cap B)}}{P(\mathbf{E})} = P(\mathbf{A}|\mathbf{E}) + P(\mathbf{B}|\mathbf{E}) - P(\mathbf{A} \cap \mathbf{B}|\mathbf{E})$$

20.
$1 = P(\mathbf{S}|\mathbf{B}) = P(\mathbf{E} \cup \mathbf{E}'|\mathbf{A}) =$

$$\frac{P[(\mathbf{E} \cup \mathbf{E}') \cap \mathbf{B}]}{P(\mathbf{B})} = \frac{P(\mathbf{E} \cap B) \cup (\mathbf{E}' \cap \mathbf{B}))}{P(\mathbf{B})} = \frac{P(\mathbf{E} \cap B) + (\mathbf{E}' \cap \mathbf{B})}{P(\mathbf{B})} = \frac{P(\mathbf{E} \cap B)}{P(\mathbf{B})} + \frac{(\mathbf{E}' \cap \mathbf{B})}{P(\mathbf{B})} =$$

$P(\mathbf{E}|\mathbf{B}) + P(\mathbf{E}'|\mathbf{B})$.

Therefore, $P(\mathbf{E}'|\mathbf{B}) = 1 - P(\mathbf{E}|\mathbf{B})$.

21.
$P(\mathbf{K_2}|\mathbf{K_1}') = 3/51$

$P(\mathbf{K_2}|\ \mathbf{K_1}) = 4/51$

$P(\mathbf{K_2}|\ \mathbf{K_1}') = 3/51 \neq 1 - P(\mathbf{K}_2|\mathbf{K}_1) = 1 - 4/51 = 47/51$

22.
$P(\mathbf{J_2}|\mathbf{K}_1 \cup \mathbf{Q_1}) = 4/51$

$P(\mathbf{J_2}|\mathbf{K}_1) + P(\mathbf{J_2}|\mathbf{Q_1}) = 4/51 + 4/51 = 8/51$

Assume an experiment generates a sample space **S** with non-empty events **A** and **B** and **A**, **B** ≠ **S.** For problems 23 - 29 select the correct answers.

23.
$P(\mathbf{A} \cap \mathbf{S}) = P(\mathbf{A})$

$P(\mathbf{A})P(\mathbf{S}) = P(\mathbf{A})(1) = P(\mathbf{A})$

Therefore, $P(\mathbf{A} \cap \mathbf{S}) = P(\mathbf{A}) = P(\mathbf{A})P(\mathbf{S})$.

The events **S** and **A** are independent.

24.
$P(\mathbf{A} \cap \mathbf{A}') = P(\phi) = 0$.

$P(\mathbf{A})P(\mathbf{B}) \neq 0$

Therefore, $P(\mathbf{A} \cap \mathbf{A}') \neq P(\mathbf{A})P(\mathbf{B})$.

The events **B** and **A** are not independent.

25.
Since **A**∩**B** = ϕ**,** if the event **A** occurs, the event **B** cannot occur. Therefore, these two events are not independent.

26.
If the event **A** occurs, then the event **B** also occurs since **A** is a subset of **B.**

27.
Not enough information.

28.

$$P(\mathbf{A}|\mathbf{B}) = \frac{P(\mathbf{A}\cap\mathbf{B})}{P(\mathbf{B})}$$

$$P(\mathbf{B}|\,\mathbf{A}) = \frac{P(\mathbf{A}\cap\mathbf{B})}{P(\mathbf{A})}$$

Since $P(\mathbf{A}|\mathbf{B}) = \mathbf{P(B|}\mathbf{A}) = \frac{P(\mathbf{A}\cap\mathbf{B})}{P(\mathbf{B})} = \frac{P(\mathbf{A}\cap\mathbf{B})}{P(\mathbf{A})}$.

Cancel $P(\mathbf{A}\cap\mathbf{B})$ in the above equation and we have $P(\mathbf{A}) = P(\mathbf{B})$.

29.

$$P(\mathbf{A}|\mathbf{B}) = \frac{P(\mathbf{A}\cap\mathbf{B})}{P(\mathbf{B})}$$

$$P(\mathbf{B}|\mathbf{A}) = \frac{P(\mathbf{A}\cap\mathbf{B})}{P(\mathbf{A})}$$

Since $P(\mathbf{A}) = P(\mathbf{B})$, $P(\mathbf{B}|\mathbf{A}) = \frac{P(\mathbf{L}'\cap\mathbf{A}')}{P(\mathbf{A}')} = P(\mathbf{A}|\mathbf{B})$.

30.
$P(\mathbf{A}\cap\mathbf{B}\cap\mathbf{C}) = P[(\mathbf{A}\cap\mathbf{B})\cap\mathbf{C}] = P(\mathbf{A}\cap\mathbf{B})P(\mathbf{C}|\mathbf{A}\cap\mathbf{B})$

$P(\mathbf{A}\cap\mathbf{B}) = P(\mathbf{A})P(\mathbf{B}|\mathbf{A})$

$P(\mathbf{A}\cap\mathbf{B}\cap\mathbf{C}) = P[(\mathbf{A}\cap\mathbf{B})\cap\mathbf{C}] = P(\mathbf{A}\cap\mathbf{B})P(\mathbf{C}|\mathbf{A}\cap\mathbf{B}) = P(\mathbf{A})P(\mathbf{B}|\mathbf{A})P(\mathbf{C}|\mathbf{A}\cap\mathbf{B})$

31.
$\mathbf{D}_1$: Event a diamond is drawn on the first drawing.
$\mathbf{F}_2$: Event a face card is drawn on the second drawing.
E: Event that a diamond and a face card are drawn.
$\mathbf{E} = \mathbf{D}_1\cap\mathbf{F}_2$

$\mathbf{D}_1 = (\mathbf{D}_1\cap\mathbf{F}_1)\cup(\mathbf{D}_1\cap\mathbf{F}_1{}')$

$\mathbf{E} = \mathbf{D_1} \cap \mathbf{F_2} = [(\mathbf{D_1} \cap \mathbf{F_1}) \cup (\mathbf{D_1} \cap \mathbf{F_1}')] \cap \mathbf{F_2} = [(\mathbf{D_1} \cap \mathbf{F_1}) \cap \mathbf{F_2}] \cup [(\mathbf{D_1} \cap \mathbf{F_1}') \cap \mathbf{F_2}]$

$P(\mathbf{E}) = P\{[(\mathbf{D_1} \cap \mathbf{F_1}) \cap \mathbf{F_2}] \cup [(\mathbf{D_1} \cap \mathbf{F_1}') \cap \mathbf{F_2}]\} = P[(\mathbf{D_1} \cap \mathbf{F_1}) \cap \mathbf{F_2}] + P[(\mathbf{D_1} \cap \mathbf{F_1}') \cap \mathbf{F_2}] =$

$$P[(\mathbf{D_1} \cap \mathbf{F_1})P[\mathbf{F_2} | (\mathbf{D_1} \cap \mathbf{F_1})] + P[(\mathbf{D_1} \cap \mathbf{F_1}')P[\mathbf{F_2} | (\mathbf{D_1} \cap \mathbf{F_1}')] = (\frac{3}{52})(\frac{11}{51}) + (\frac{10}{52})(\frac{12}{51}) = \frac{51}{884}$$

32.
Step 1: $\mathbf{D_1}$: first card drawn is a diamond.

$\mathbf{K_1}$: first card drawn is a king.

$\mathbf{D_2}$: second card drawn is a diamond.

Step 2: $\mathbf{D_2} \cap \mathbf{K_1} = [(\mathbf{D_2} \cap \mathbf{K_1}) \cap \mathbf{D_1}] \cup [(\mathbf{D_2} \cap \mathbf{K_1}) \cap \mathbf{D_1}'] = [\mathbf{D_2} \cap (\mathbf{K_1} \cap \mathbf{D_1})] \cup [\mathbf{D_2} \cap (\mathbf{K_1} \cap \mathbf{D_1}')]$

Step 3: $P(\mathbf{D_2} \cap \mathbf{K_1}) = P\{[(\mathbf{D_2} \cap \mathbf{K_1}) \cap \mathbf{D_1}] \cup [(\mathbf{D_2} \cap \mathbf{K_1}) \cap \mathbf{D_1}'] = P\{[\mathbf{D_2} \cap (\mathbf{K_1} \cap \mathbf{D_1})] \cup [\mathbf{D_2} \cap (\mathbf{K_1} \cap \mathbf{D_1}')]\} =$

$= P[\mathbf{D_2} \cap (\mathbf{K_1} \cap \mathbf{D_1})] + P[\mathbf{D_2} \cap (\mathbf{K_1} \cap \mathbf{D_1}')]\} = P[\mathbf{D_2} | (\mathbf{K_1} \cap \mathbf{D_1})]P(\mathbf{K_1} \cap \mathbf{D_1}) + P[\mathbf{D_2} | (\mathbf{K_1} \cap \mathbf{D_1}')]P(\mathbf{K_1} \cap \mathbf{D_1}') =$

$$(\frac{12}{51})(\frac{1}{52}) + (\frac{13}{51})(\frac{3}{52}) = \frac{1}{52}$$

Step 4: $$P(\mathbf{D_2} | \mathbf{k_1}) = \frac{P(\mathbf{K_1} \cap \mathbf{D_2})}{P(\mathbf{K_1})} = \frac{(\frac{1}{52})}{(\frac{4}{52})} = \frac{1}{4}$$

33.
$P(\mathbf{E_1})P(\mathbf{E_2} | \mathbf{E_1})P(\mathbf{E_3} | \mathbf{E_1} \cap \mathbf{E_2})P(\mathbf{E_4} | \mathbf{E_1} \cap \mathbf{E_2} \cap \mathbf{E_3}) \ldots P(\mathbf{E_n} | \mathbf{E_1} \cap \mathbf{E_2} \cap \mathbf{E_3} \ldots \cap \mathbf{E_{n-1}}) =$

$$P(\mathbf{E_1}) \frac{P(\mathbf{E_1} \cap \mathbf{E_2})}{P(\mathbf{E_1})} \frac{P(\mathbf{E_1} \cap \mathbf{E_2} \cap \mathbf{E_3})}{P(\mathbf{E_1} \cap \mathbf{E_2})} \frac{P(\mathbf{E_1} \cap \mathbf{E_2} \cap \mathbf{E_3} \cap \mathbf{E_4})}{P(\mathbf{E_1} \cap \mathbf{E_2} \cap \mathbf{E_3})} \ldots$$

$$\frac{P(\mathbf{E_1} \cap \mathbf{E_2} \cap \mathbf{E_3} \cap \mathbf{E_4} \ldots \cap \mathbf{E_{n-1}})}{P(\mathbf{E_1} \cap \mathbf{E_2} \cap \mathbf{E_3} \ldots \cap \mathbf{E_{n-2}})} \frac{P(\mathbf{E_1} \cap \mathbf{E_2} \cap \mathbf{E_3} \cap \mathbf{E_4} \ldots \cap \mathbf{E_n})}{P(\mathbf{E_1} \cap \mathbf{E_2} \cap \mathbf{E_3} \ldots \cap \mathbf{E_{n-1}})} = P(\mathbf{E_1} \cap \mathbf{E_2} \cap \mathbf{E_3} \ldots \cap \mathbf{E_n})$$

34.
Step 1: $\#\mathbf{S} = 36$

$\#\mathbf{E_1} = (3)(6) = 18$

$\#\mathbf{E_2} = (3)(6) = 18$

$\mathbf{B} = \{(1,2),(1,4),(1,6),(2,1),(2,3),(2,5),(3,2),(3,4),(3,6),(4,1),(4,3),(4,5),(5,2),(5,4),(5,6),$
$(6,1),(6,3),(6,5)\}$

$\#\mathbf{B} = 18$

$\mathbf{E_1}\cap\mathbf{E_2} = \{(2,2),(2,4),(2,6),(4,2),(4,4),(4,6),(6,2),(6,4),(6,6)\}$

$\#\mathbf{E_1}\cap\mathbf{E_2} = 9$

$\mathbf{E_1}\cap\mathbf{B} = \{(2,1),(2,3),(2,5),(4,1),(4,3),(4,5),(6,1),(6,3),(6,5)\}$

$\#\mathbf{E_1}\cap\mathbf{B} = 9$

$\mathbf{E_2}\cap\mathbf{B} = \{(1,2),(3,2),(5,2),(1,4),(3,4),(5,4),(5,6),(5,6),(5,6)\}$

$\#\mathbf{E_2}\cap\mathbf{B} = 9$

Step 2: $P(\mathbf{E_1}) = 18/36 = 1/2$

$P(\mathbf{E_2}) = 18/36 = 1/2$

$P(\mathbf{B}) = 18/36 = 1/2$

$P(\mathbf{E_1}\cap\mathbf{E_2}) = 9/36 = 1/4 = P(\mathbf{E_1})P(\mathbf{E_2})$

$P(\mathbf{E_1}\cap\mathbf{B}) = 9/36 = 1/4 = P(\mathbf{E_1})P(\mathbf{B})$

$P(\mathbf{E_2}\cap\mathbf{B}) = 9/36 = 1/4 = P(\mathbf{E_2})P(\mathbf{B})$

Therefore, these three events are pair-wise independent.

Step 3: The only way the sum of two numbers can be odd is that one number must be even and the other number must be odd. The event $\mathbf{E_1}\cap\mathbf{E_2}\cap\mathbf{B}$ means that both numbers are even and the sum is odd. Since this is impossible,

$\mathbf{E_1}\cap\mathbf{E_2}\cap\mathbf{B} = \phi$

Therefore, $P(\mathbf{E_1}\cap\mathbf{E_2}\cap\mathbf{B}) = P(\phi) = 0 \neq P(\mathbf{E_1})P(\mathbf{E_2})P(\mathbf{B}) = 1/8$.

The events are not mutually independent.

35.
Step 1: #**S** = 216

$\mathbf{E}_{1,2}$ = {(1,1,1),(1,1,2),(1,1,3),(1,1,4),(1,1,5),(1,1,6),
(2,2,1),(2,2,2),(2,2,3),(2,2,4),(2,2,5),(2,2,6),
(3,3,1),(3,3,2),(3,3,3),(3,3,4),(3,3,5),(3,3,6),
(4,4,1),(4,4,2),(4,4,3),(4,4,4),(4,4,5),(4,4,6),
(5,5,1),(5,5,2),(5,5,3),(5,5,4),(5,5,5),(5,5,6),
(6,6,1),(6,6,2),(6,6,3),(6,6,4),(6,6,5),(6,6,6)}

$\#\mathbf{E}_{1,2} = 36$

In a similar way we can show that
$\#\mathbf{E}_{1,3} = 36$

$\#\mathbf{E}_{2,3} = 36$

$\mathbf{E}_{1,2}\cap\mathbf{E}_{1,3}$ = {(1,1,1),(2,2,2),(3,3,3),(4,4,4),(5,5,5),(6,6,6)}

$\#\mathbf{E}_{1,2}\cap\mathbf{E}_{1,3} = 6$

$\mathbf{E}_{1,2}\cap\mathbf{E}_{2,3}$ = {(1,1,1),(2,2,2),(3,3,3),(4,4,4),(5,5,5),(6,6,6)}

$\#\mathbf{E}_{1,2}\cap\mathbf{E}_{2,3} = 6$

$\mathbf{E}_{1,3}\cap\mathbf{E}_{2,3}$ = {(1,1,1),(2,2,2),(3,3,3),(4,4,4),(5,5,5),(6,6,6)}

$\#\mathbf{E}_{1,3}\cap\mathbf{E}_{2,3} = 6$

Step 2: $P(\#\mathbf{E}_{1,2}) = \dfrac{36}{216} = \dfrac{1}{6}$

$P(\#\mathbf{E}_{1,3}) = \dfrac{36}{216} = \dfrac{1}{6}$

$P(\#\mathbf{E}_{2,3}) = \dfrac{36}{216} = \dfrac{1}{6}$

$P(\mathbf{E}_{1,2}\cap\mathbf{E}_{1,3}) = \dfrac{6}{216} = \dfrac{1}{36} = P(\mathbf{E}_{1,2})P(\mathbf{E}_{1,3}) = (\dfrac{36}{216})(\dfrac{36}{216}) = \dfrac{1}{36}$

$P(\mathbf{E}_{1,2} \cap \mathbf{E}_{2,3}) = \frac{6}{216} = \frac{1}{36} = P(\mathbf{E}_{1,2})P(\mathbf{E}_{2,3}) = (\frac{36}{216})(\frac{36}{216}) = \frac{1}{36}$

$P(\mathbf{E}_{1,3} \cap \mathbf{E}_{2,3}) = \frac{6}{216} = \frac{1}{36} = P(\mathbf{E}_{1,2})P(\mathbf{E}_{2,3}) = (\frac{36}{216})(\frac{36}{216}) = \frac{1}{36}$

Therefore, these events are pair-wise independent.

Step 3: $\mathbf{E}_{1,3} \cap \mathbf{E}_{2,3} \cap \mathbf{E}_{1,2} = \{(1,1,1),(2,2,2),(3,3,3),(4,4,4),(5,5,5),(6,6,6)\}$

$\#(\mathbf{E}_{1,3} \cap \mathbf{E}_{2,3} \cap \mathbf{E}_{1,2}) = 6$

$P(\mathbf{E}_{1,3} \cap \mathbf{E}_{2,3} \cap \mathbf{E}_{1,2}) = \frac{6}{216} = \frac{1}{36} \neq P(\mathbf{E}_{1,3})P(\mathbf{E}_{2,3})P(\mathbf{E}_{1,2}) = (\frac{1}{6})(\frac{1}{6})(\frac{1}{6}) = \frac{1}{216}$

Therefore, these events are not mutually independent.

36.

➤ a.

$P(\mathbf{E} \cap \mathbf{F}) = P(\mathbf{A} \cap \mathbf{B} \cap \mathbf{C} \cap \mathbf{D}) = P(\mathbf{A})P(\mathbf{B})P(\mathbf{C})P(\mathbf{D}) = [P(\mathbf{A})P(\mathbf{B})][P(\mathbf{C})P(\mathbf{D})] =$

$[P(\mathbf{A})P(\mathbf{B})][P(\mathbf{C})P(\mathbf{D})] = P(\mathbf{A} \cap \mathbf{B})P(\mathbf{C} \cap \mathbf{D}) = P(\mathbf{E})P(\mathbf{F})$

➤ b.

$P(\mathbf{E} \cap \mathbf{D}) = P\{(\mathbf{B} \cup \mathbf{C}) \cap \mathbf{D}\} = P\{(\mathbf{B} \cap \mathbf{D}) \cup (\mathbf{C} \cap \mathbf{D})\} = P(\mathbf{B} \cap \mathbf{D}) + P(\mathbf{C} \cap \mathbf{D}) - P(\mathbf{B} \cap \mathbf{C} \cap \mathbf{D}) =$

$P(\mathbf{B})P(\mathbf{D}) + P(\mathbf{C})P(\mathbf{D}) - P(\mathbf{B})P(\mathbf{C})P(\mathbf{D}) = [P(\mathbf{B}) + P(\mathbf{C}) - P(\mathbf{B})P(\mathbf{C})]P(\mathbf{D}) = P(\mathbf{B} \cup \mathbf{C})P(\mathbf{D}) =$

$P(\mathbf{E})P(\mathbf{D})$

37.

$P(\mathbf{A}_1 \cap \mathbf{A}_2) = P[\mathbf{A}_1 \cap \mathbf{A}_2 \cap \mathbf{S}] = P[\mathbf{A}_1 \cap \mathbf{A}_2 \cap (\mathbf{A}_3 \cup \mathbf{A}_3')] = P[(\mathbf{A}_1 \cap \mathbf{A}_2 \cap \mathbf{A}_3) \cup (\mathbf{A}_1 \cap \mathbf{A}_2 \cap \mathbf{A}_3')] =$

$= P(\mathbf{A}_1 \cap \mathbf{A}_2 \cap \mathbf{A}_3) + P(\mathbf{A}_1 \cap \mathbf{A}_2 \cap \mathbf{A}_3') = P(\mathbf{A}_1)P(\mathbf{A}_2)P(\mathbf{A}_3) + P(\mathbf{A}_1)P(\mathbf{A}_2)P(\mathbf{A}_3') =$

$P(\mathbf{A}_1)P(\mathbf{A}_2)[P(\mathbf{A}_3) + P(\mathbf{A}_3')] = P(\mathbf{A}_1)P(\mathbf{A}_2)(1) = P(\mathbf{A}_1)P(\mathbf{A}_2)$

By changing the subscripts in the above equations, do the same for $P(\mathbf{A}_2 \cap \mathbf{A}_3)$ and $P(\mathbf{A}_1 \cap \mathbf{A}_3)$.

Finally, $P(\mathbf{A}_1 \cap \mathbf{A}_2 \cap \mathbf{A}_3) = P(\mathbf{A}_1)P(\mathbf{A}_2)P(\mathbf{A}_3)$.

Therefore, these three events are mutually independent.

38.
Step 1: Define the following events:

$\mathbf{E}_2$: The event exactly 2 cards are drawn .
$\mathbf{E}_3$: The event exactly 3 cards are drawn.
$\mathbf{E}_4$: The event exactly 4 cards are drawn. .
$\mathbf{E}$ = the event that 5 cards were drawn
$\mathbf{E}'$ = the event that less than 5 cards were drawn.
$\mathbf{E}' = \mathbf{E}_2 \cup \mathbf{E}_3 \cup \mathbf{E}_4$

Step 2: There are 13 diamonds and 39 non-diamonds in an ordinary deck of cards.

$\mathbf{D}_k$: The event that a diamond is drawn on the kth drawing (k = 1,2,3,4)

$\mathbf{E}_2 = \mathbf{D}_1 \cap \mathbf{D}_2$

$\mathbf{E}_3 = (\mathbf{D}_1 \cap \mathbf{D}_2' \cap \mathbf{D}_3) \cup (\mathbf{D}_1' \cap \mathbf{D}_2 \cap \mathbf{D}_3)$

$\mathbf{E}_4 = (\mathbf{D}_1' \cap \mathbf{D}_2' \cap \mathbf{D}_3 \cap \mathbf{D}_4) \cup (\mathbf{D}_1' \cap \mathbf{D}_2 \cap \mathbf{D}_3' \cap \mathbf{D}_4) \cup (\mathbf{D}_1 \cap \mathbf{D}_2' \cap \mathbf{D}_3' \cap \mathbf{D}_4)$

$P(\mathbf{E}') = P(\mathbf{E}_2) + P(\mathbf{E}_3) + P(\mathbf{E}_4) = P(\mathbf{D}_1 \cap \mathbf{D}_2) + P(\mathbf{D}_1 \cap \mathbf{D}_2' \cap \mathbf{D}_3) + P(\mathbf{D}_1' \cap \mathbf{D}_2 \cap \mathbf{D}_3) +$

$+ P(\mathbf{D}_1' \cap \mathbf{D}_2' \cap \mathbf{D}_3 \cap \mathbf{D}_4)) + P(\mathbf{D}_1' \cap \mathbf{D}_2 \cap \mathbf{D}_3' \cap \mathbf{D}_4) + P(\mathbf{D}_1 \cap \mathbf{D}_2' \cap \mathbf{D}_3' \cap \mathbf{D}_4) =$

$$(\frac{13}{52})(\frac{12}{51}) + (\frac{13}{52})(\frac{39}{51})(\frac{12}{50}) + (\frac{39}{52})(\frac{13}{51})(\frac{12}{50}) + (\frac{39}{52})(\frac{38}{51})(\frac{13}{50})(\frac{12}{49}) + (\frac{39}{52})(\frac{13}{51})(\frac{38}{50})(\frac{12}{49}) + (\frac{13}{52})(\frac{39}{51})(\frac{38}{50})(\frac{12}{49})$$

$$+ (\frac{13}{52})(\frac{12}{51}) + 2(\frac{13}{52})(\frac{39}{51})(\frac{12}{50}) + 3(\frac{39}{52})(\frac{38}{51})(\frac{13}{50})(\frac{12}{49}) = \frac{156}{2652} + \frac{12168}{132600} + \frac{693576}{6497400} = \frac{1672008}{6497400}$$

$$P(\mathbf{E}) = 1 - P(\mathbf{E}') = 1 - \frac{1672008}{6497400} = \frac{4825392}{6497400}$$

39.
$\mathbf{K}_1'$: The first card is not a king.
$\mathbf{Q}_1$: The first card is a queen.
$\mathbf{J}_1$**:** The first card is a jack.
$\mathbf{F}_1'$**:** The first card is not a face card.

$$P(\mathbf{F}_2 \mid \mathbf{K}_1') = \frac{P(\mathbf{F_2} \cap \mathbf{K_1}')}{P(\mathbf{K_1}')} \quad \frac{P(\mathbf{F_2} \cap \mathbf{K_1}')}{\frac{48}{52}} = P(\mathbf{F_2} \cap \mathbf{K_1}')(\frac{13}{12})$$

$\mathbf{K_1}' = \mathbf{Q_1} \cup \mathbf{J_1} \cup \mathbf{F_1}'$

$\mathbf{F}_2 \cap \mathbf{K_1}' = \mathbf{F}_2 \cap (\mathbf{Q_1} \cup \mathbf{J}_1 \cup \mathbf{F_1}') = (\mathbf{F}_2 \cap \mathbf{Q_1}) \cup (\mathbf{F}_2 \cap \mathbf{J_1}) \cap (\mathbf{F}_2 \cap \mathbf{F}_1')$

$P(\mathbf{F_2} \cap \mathbf{K}_1') = P\mathbf{F}_2 \cap \mathbf{Q_1}) + P\mathbf{F}_2 \cap \mathbf{J_1}) + P(\mathbf{F_2} \cap \mathbf{F_1}') = P(\mathbf{Q}_1)P(\mathbf{F}_2 | \mathbf{Q_1}) + P(\mathbf{J}_1)P(\mathbf{F}_2 | \mathbf{J_1}) + P(\mathbf{F}_1')P(\mathbf{F}_2 | \mathbf{F_1}')$

$= (1/13)(11/51) + (1/13)(11/51) + (10/13)(12/51)$

$P(\mathbf{F}_2 \cap \mathbf{K}_1') = [(1/13)(11/51) + (1/13)(11/51) + (10/13)(12/51)](13/12) = 142/612$

$$[(\frac{1}{13})(\frac{11}{51}) + (\frac{1}{13})(\frac{11}{51}) + (\frac{10}{13})(\frac{12}{51})](\frac{13}{12}) = \frac{142}{612}$$

40.
K: The event a king is drawn.
D: The event a diamond is drawn.
E: An arbitrary non-trivial event ($0 < P(\mathbf{E}) < 1$)

Assume the three events are mutually independent.

$P(\mathbf{K}) = 1/13$

$P(\mathbf{D}) = 1/4$

$P(\mathbf{E}) = n/52$, where $0 < n < 52$.

$P(\mathbf{K} \cap \mathbf{D} \cap \mathbf{E}) = m/52$, where $0 < m$.

If these three events are mutually independent then the following must hold:

$P(\mathbf{K} \cap \mathbf{D} \cap \mathbf{E}) = P(\mathbf{K})P(\mathbf{D}) P(\mathbf{E}) =(1/13)(1/4)(n/52) = m/52.$

Simplifying this equation give us $n = 52m$.

This contradicts the condition above that $0 < n < 52$.

41.
➤ a.
S = {(h,h), (h,t,h), (t,h,h), (t,t,t,t), (h,t,t,t), (t,h,t,t), {t,t,h,t), (t,t,t,h), (t,t,h,h), {t,h,t,h), (h,t,t,h)}

Because the tosses are independent,

P{(h,h)} = (1/2)(1/2) = 1/4, P{(h,t,h)} (1/2)(1/2)(1/2) = 1/8, P{(t,h,h)} = 1/8,

P{(t,t,t,t)} = 1/16, P{(h,t,t,t)} = 1/16, P{(t,h,t,t)} = 1/16 P{(t,t,h,t)} = 1/16

P{(t,t,t,h)} = 1/16 P{(t,t,h,h)} = 1/16, P{(t,h,t,h)} = 1/16, P{(h,t,t,h)} = 1/16

The sample space is

S = {(h,h), (h,t,h), (t,h,h), (t,t,t,t), (h,t,t,t), (t,h,t,t), {t,t,h,t), (t,t,t,h)), (t,t,h,h), {t,h,t,h), (h,t,t,h)}

➤b.
Heads on the first toss: $\mathbf{H}_1$ = {(h,h), (h,t,h), (h,t,t,t), (h,t,t,h)}.

Heads on the fourth toss: $\mathbf{H}_4$ = {(t,t,t,h)), (t,t,h,h), {t,h,t,h), (h,t,t,h)}

P{$\mathbf{H}_1$} = (1/2)(1/2) + (1/2)(1/2)(1/2) + (1/2)(1/2)(1/2)(1/2) + (1/2)(1/2)(1/2)(1/2) = 1/2.

P{$\mathbf{H}_4$} = (1/2)(1/2)(1/2)(1/2) + (1/2)(1/2)(1/2)(1/2) + (1/2)(1/2)(1/2)(1/2) + (1/2)(1/2)(1/2)(1/2)

= 1/4

$\mathbf{H}_1 \cap \mathbf{H}_4$ = {(h,t,t,h)}

P{$\mathbf{H}_1 \cap \mathbf{H}_4$ } = 1/16

P{$\mathbf{H}_1$ }P{$\mathbf{H}_4$ } = (1/2)(1/4) = 1/8

Therefore, P{$\mathbf{H}_1 \cap \mathbf{H}_4$ } ≠ P{$\mathbf{H}_1$ }P{$\mathbf{H}_4$ }.

This means that the $\mathbf{H}_1$ and $\mathbf{H}_2$ and **NOT** independent.

➤c.
$\mathbf{H}_k$: The event that the k th toss is a head.
$\mathbf{T}_k$: The event that the k th toss is a tail.
F: The event that 4 tosses occurred.

$\mathbf{F} = (\mathbf{T}_1 \cap \mathbf{T}_2 \cap \mathbf{H}_3 \cap \mathbf{H}_3) \cup (\mathbf{T}_1 \cap \mathbf{H}_2 \cap \mathbf{T}_3 \cap \mathbf{H}_3) \cup (\mathbf{H}_1 \cap \mathbf{T}_2 \cap \mathbf{T}_3 \cap \mathbf{H}_3) \cup (\mathbf{T}_1 \cap \mathbf{T}_2 \cap \mathbf{T}_3 \cap \mathbf{H}_3) \cup (\mathbf{T}_1 \cap \mathbf{T}_2 \cap \mathbf{H}_3 \cap \mathbf{T}_3) \cup$

$(\mathbf{T}_1 \cap \mathbf{H}_2 \cap \mathbf{T}_3 \cap \mathbf{T}_3) \cup (\mathbf{H}_1 \cap \mathbf{T}_2 \cap \mathbf{T}_3 \cap \mathbf{T}_3) \cup (\mathbf{T}_1 \cap \mathbf{T}_2 \cap \mathbf{T}_3 \cap \mathbf{T}_3)$

$P(\mathbf{F}) = P(\mathbf{T}_1 \cap \mathbf{T}_2 \cap \mathbf{H}_3 \cap \mathbf{H}_3) + P(\mathbf{T}_1 \cap \mathbf{H}_2 \cap \mathbf{T}_3 \cap \mathbf{H}_3) + P(\mathbf{H}_1 \cap \mathbf{T}_2 \cap \mathbf{T}_3 \cap \mathbf{H}_3) + P\ (\mathbf{T}_1 \cap \mathbf{T}_2 \cap \mathbf{T}_3 \cap \mathbf{H}_3) +$

P $(\mathbf{T}_1\cap\mathbf{T}_2\cap\mathbf{H}_3\cap\mathbf{T}_3)$ + P$(\mathbf{T}_1\cap\mathbf{H}_2\cap\mathbf{T}_3\cap\mathbf{T}_3)$ + P$(\mathbf{H}_1\cap\mathbf{T}_2\cap\mathbf{T}_3\cap\mathbf{T}_3)$ + P$(\mathbf{T}_1\cap\mathbf{T}_2\cap\mathbf{T}_3\cap\mathbf{T}_3)$ =

1/16 + 1/16 + 1/16 + 1/16 + 1/16 + 1/16 + 1/16 + 1/16 = 1/2.

➤d.
$\mathbf{T}_4$: The event that 4 tosses occurred.
$\mathbf{H}_1$: The event that a head occurred on the first toss.

We need to find $P(\mathbf{H_1}|\mathbf{T}_4)$.

$$P(\mathbf{H_1}|\mathbf{T_4}) = \frac{P(\mathbf{H_1}\cap\mathbf{T_4})}{P(\mathbf{T_4})}$$

$\mathbf{H}_1\cap\mathbf{T}_4$ = {(h,t,t,h), (h,t,t,t) }

$P(\mathbf{H}_1\cap\mathbf{T}_4)$ = P({(h,t,t,h), (h,t,t,t) }) = 1/16 + 1/16 = 1/8.

From (d), $P(\mathbf{T}_4) = 1/2$.

$$P(\mathbf{H_1}|\mathbf{T_4}) = \frac{P(\mathbf{H_1}\cap\mathbf{T_4})}{P(\mathbf{T_4})} = \frac{\frac{1}{8}}{\frac{1}{2}} = \frac{1}{4}.$$

42.
➤a.
E: the event that a one card is a king and the other card is any kind of diamond.

Since order is not required we must consider all posibilities: $\mathbf{E} = (\mathbf{K}_1\cap\mathbf{D}_2)\cup(\mathbf{D}_1\cap\mathbf{K}_2)$

$\mathbf{K}_1 = (\mathbf{K}_1\cap\mathbf{D}_1)\cup(\mathbf{K}_1\cap\mathbf{D}_1')$

$\mathbf{D}_1 = (\mathbf{D}_1\cap\mathbf{K}_1)\cup(\mathbf{D}_1\cap\mathbf{K}_1')$

$\mathbf{K}_1\cap\mathbf{D}_2 = [(\mathbf{K}_1\cap\mathbf{D}_1)\cup(\mathbf{K}_1\cap\mathbf{D}_1')]\cap\mathbf{D}_2 = (\mathbf{K}_1\cap\mathbf{D}_1\cap\mathbf{D}_2)\cup(\mathbf{K}_1\cap\mathbf{D}_1'\cap\mathbf{D}_2)$

$\mathbf{D}_1\cap\mathbf{K}_2 = [(\mathbf{D}_1\cap\mathbf{K}_1)\cup(\mathbf{D}_1\cap\mathbf{K}_1')]\cap\mathbf{K}_2 = (\mathbf{D}_1\cap\mathbf{K}_1\cap\mathbf{K}_2)\cup(\mathbf{D}_1\cap\mathbf{K}_1'\cap\mathbf{K}_2)$

$\mathbf{E} = [(\mathbf{K}_1\cap\mathbf{D}_1\cap\mathbf{D}_2)\cup(\mathbf{K}_1\cap\mathbf{D}_1'\cap\mathbf{D}_2)]\cup[(\mathbf{D}_1\cap\mathbf{K}_1\cap\mathbf{K}_2)\cup(\mathbf{D}_1\cap\mathbf{K}_1'\cap\mathbf{K}_2)]$

$P(\mathbf{E}) = P(\mathbf{K}_1\cap\mathbf{D}_2) + P(\mathbf{D}_1\cap\mathbf{K}_2)$ =

$P[(\mathbf{K}_1\cap\mathbf{D}_1\cap\mathbf{D}_2) + P(\mathbf{K}_1\cap\mathbf{D}_1'\cap\mathbf{D}_2) + P(\mathbf{D}_1\cap\mathbf{K}_1\cap\mathbf{K}_2) + P(\mathbf{D}_1\cap\mathbf{K}_1'\cap\mathbf{K}_2)$ =

(1/52)(12/51) + (3/52)(13/51) + (1/52)(3/51) + (12/52)(4/51) =

$$(\frac{1}{52})(\frac{12}{51}) + (\frac{3}{52})(\frac{13}{51}) + (\frac{1}{52})(\frac{3}{51}) + (\frac{12}{52})(\frac{4}{51}) = \frac{102}{2652}$$

➤b.
We use the formula (#**S**)P(**E**) = #**E**

#**S** = 2652

$P(\mathbf{E}) = \frac{102}{2652}$

Therefore, $\#\mathbf{E} = 2652(\frac{102}{2652}) = 102$.

43.

$$P(\mathbf{K}_1 \cap \mathbf{K}_2) = (\frac{4}{52})(\frac{3}{51}) = \frac{12}{2652}$$

$$P(\mathbf{K}_1 \cap \mathbf{K}_3) = P[(\mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_3) \cup (\mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_3')] = P(\mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_3) + P(\mathbf{K}_1 \cap \mathbf{K}_2' \cap \mathbf{K}_3) =$$

$$(\frac{4}{52})(\frac{3}{51})(\frac{2}{50}) + (\frac{4}{52})(\frac{48}{51})(\frac{3}{50}) = \frac{12}{2652}$$

$$P(\mathbf{K}_2 \cap \mathbf{K}_3) = P[(\mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_3) \cup (\mathbf{K}_1' \cap \mathbf{K}_2 \cap \mathbf{K}_3)] = P(\mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_3) + P(\mathbf{K}_1' \cap \mathbf{K}_2 \cap \mathbf{K}_3) =$$

$$(\frac{4}{52})(\frac{3}{51})(\frac{2}{50}) + (\frac{48}{52})(\frac{4}{51})(\frac{3}{50}) = \frac{12}{2652}$$

44.
➤a.
E is the event that the first three cards have at least 1 king and the last 2 cards drawn have no kings.

➤b.

$$\mathbf{E} = (\mathbf{K}_1 \cup \mathbf{K}_2 \cup \mathbf{K}_3) \cap (\mathbf{K}_4' \cap \mathbf{K}_5') = (\mathbf{K}_1 \cap \mathbf{K}_4' \cap \mathbf{K}_5') \cup (\mathbf{K}_2 \cap \mathbf{K}_4' \cap \mathbf{K}_5') \cup (\mathbf{K}_3 \cap \mathbf{K}_4' \cap \mathbf{K}_5')$$

$$P(\mathbf{E}) = P[(\mathbf{K}_1 \cap \mathbf{K}_4' \cap \mathbf{K}_5') \cup (\mathbf{K}_2 \cap \mathbf{K}_4' \cap \mathbf{K}_5') \cup (\mathbf{K}_3 \cap \mathbf{K}_4' \cap \mathbf{K}_5')]$$

To solve this, we will use problem 43 and the following identity:

$$P(\mathbf{A} \cup \mathbf{B} \cup \mathbf{C}) = P(\mathbf{A}) + P(\mathbf{B}) + P(\mathbf{C}) - P(\mathbf{A} \cap \mathbf{B}) - P(\mathbf{A} \cap \mathbf{C}) - P(\mathbf{B} \cap \mathbf{C}) + P(\mathbf{A} \cap \mathbf{B} \cap \mathbf{C})$$

Let

$\mathbf{A} = (\mathbf{K}_1 \cap \mathbf{K}_4' \cap \mathbf{K}_5')$

$\mathbf{B} = (\mathbf{K}_2 \cap \mathbf{K}_4' \cap \mathbf{K}_5')$

$\mathbf{C} = (\mathbf{K}_3 \cap \mathbf{K}_4' \cap \mathbf{K}_5')$

$P(\mathbf{E}) = P(\mathbf{K}_1 \cap \mathbf{K}_4' \cap \mathbf{K}_5') + P(\mathbf{K}_2 \cap \mathbf{K}_4' \cap \mathbf{K}_5') + P(\mathbf{K}_3 \cap \mathbf{K}_4' \cap \mathbf{K}_5') -$

$P[(\mathbf{K}_1 \cap \mathbf{K}_4' \cap \mathbf{K}_5') \cap (\mathbf{K}_2 \cap \mathbf{K}_4' \cap \mathbf{K}_5')] - P[(\mathbf{K}_1 \cap \mathbf{K}_4' \cap \mathbf{K}_5') \cap (\mathbf{K}_3 \cap \mathbf{K}_4' \cap \mathbf{K}_5')] -$

$P[(\mathbf{K}_2 \cap \mathbf{K}_4' \cap \mathbf{K}_5'\} \cap (\mathbf{K}_3 \cap \mathbf{K}_4' \cap \mathbf{K}_5')] \ -P[(\mathbf{K}_1 \cap \mathbf{K}_4' \cap \mathbf{K}_5') \cap (\mathbf{K}_2 \cap \mathbf{K}_4' \cap \mathbf{K}_5') \cap (\mathbf{K}_3 \cap \mathbf{K}_4' \cap \mathbf{K}_5')] =$

$P(\mathbf{K}_1 \cap \mathbf{K}_4' \cap \mathbf{K}_5') + P(\mathbf{K}_2 \cap \mathbf{K}_4' \cap \mathbf{K}_5') + P(\mathbf{K}_3 \cap \mathbf{K}_4' \cap \mathbf{K}_5') - P(\mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_4' \cap \mathbf{K}_5') -$

$P(\mathbf{K}_1 \cap \mathbf{K}_3 \cap \mathbf{K}_4' \cap \mathbf{K}_5') - P(\mathbf{K}_2 \cap \mathbf{K}_3 \cap \mathbf{K}_4' \cap \mathbf{K}_5') + P(\mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_3 \cap \mathbf{K}_4' \cap \mathbf{K}_5')] =$

$$3\left(\frac{4}{52}\right)\left(\frac{48}{51}\right)\left(\frac{47}{50}\right) - 3\left(\frac{4}{52}\right)\left(\frac{3}{51}\right)\left(\frac{48}{50}\right)\left(\frac{47}{49}\right) + \left(\frac{4}{52}\right)\left(\frac{3}{51}\right)\left(\frac{2}{50}\right)\left(\frac{48}{49}\right)\left(\frac{47}{48}\right) = \frac{5982912}{31187520}$$

➤c.
$\#\mathbf{S}P(\mathbf{E}) = \#\mathbf{E}$

$\#\mathbf{E} = (311{,}875{,}200)(59{,}829{,}120)/(311{,}875{,}200) = 59{,}829{,}120.$

45.
➤ a.
$\mathbf{K}_1$: The event hand 1 has exactly 1 king.
$\mathbf{K}_2$: The event hand 2 has exactly 1 king.
$\mathbf{K}_1 \cap \mathbf{K}_2$: The event both hands have exactly 1 king.

$$P(\mathbf{K}_1) = 2\left(\frac{4}{52}\right)\left(\frac{48}{51}\right) = \frac{384}{2652}$$

$$P(\mathbf{K}_2 | \mathbf{K}_1) = 2\left(\frac{3}{50}\right)\left(\frac{47}{49}\right) = \frac{282}{2450}$$

$$P(\mathbf{K}_1 \cap \mathbf{K}_2) = P(\mathbf{K}_1)P(\mathbf{K}_2 | \mathbf{K}_1) = \left(\frac{384}{2652}\right)\left(\frac{282}{2450}\right) = \frac{108288}{6497400}.$$

➤ b.
$\mathbf{K}_1 \cup \mathbf{K}_2$: The event that at least 1 hand has exactly 1 king.

$P(\mathbf{K}_1 \cup \mathbf{K}_2) = P(\mathbf{K}_1) + P(\mathbf{K}_2) - P(\mathbf{K}_1 \cap \mathbf{K}_2) = \frac{384}{2652} + \frac{384}{2652} - \frac{108288}{6497400} = \frac{1773312}{6497400}$.

46.

➤ a.

A: the event that the sum is six.

B: the event that the first toss resulted in a 2.

We need to find P(**B**|**A**) = #(**A**∩**B**)/ #**A**

A = {(1,5), (2,4), (3,3), (4,2), (5,1)}

B = {(2,1), (2,2), (2,3), (2,3), (2,4), (2,5), (2,6)}

A∩**B** = {(2,4)}

#**A** = 5

(**A**∩**B**) = 1

Therefore, P(**B**|**A**) = #(**A**∩**B**)/ #**A** = 1/5.

➤ b.

C: the event that the sum is five.

We need to find P(**B**|**A**∪**C**) = #[(**A**∪**C**)∩**B**)]/ #(**A**∪**C**) = #[(**A**∩**B**)∪(**C**∩**B**))]/ #(**A**∪**C**) =

[#(**A**∩**B**) + #(**C**∩**B**))]/[#**A** + #**C**] .

C: {(1,4), (2,3), (3,2), (4,1)}

#(**A**∩**B**) = 1

#(**C**∩**B**) = 1

#**A** = 5

#**B** = 4

Therefore, P(**B**|**A**∪**C**) = [1 + 1]/ [5 + 4] = 2/9

47.

$\mathbf{U}_A$: The event urn A is selected.

$\mathbf{U}_B$: The event urn B is selected.

$\mathbf{R}_1$: The event the first marble selected is red.
$\mathbf{W}_1$: The event the first marble selected is white.
$\mathbf{R}_2$: The event the second marble selected is red.

$P(\mathbf{U}_A) = 1/4$

$P(\mathbf{U}_B) = 3/4$

$P(\mathbf{R}_1|\mathbf{U}_A) = 6/9$
$P(\mathbf{W}_1|\mathbf{U}_A) = 3/9$

$P(\mathbf{R}_1|\mathbf{U}_B) = 8/19$
$P(\mathbf{W}_1|\mathbf{U}_B) = 11/19$

$P(\mathbf{R}_2|\mathbf{U}_A\cap\mathbf{R}_1) = 9/20$
$P(\mathbf{R}_2|\mathbf{U}_A\cap\mathbf{W}_1) = 8/20$

$P(\mathbf{R}_2|\mathbf{U}_B\cap\mathbf{R}_1) = 7/10$
$P(\mathbf{R}_2|\mathbf{U}_B\cap\mathbf{W}_1) = 6/10$

$\mathbf{R}_2 = \{\mathbf{U}_A\cap[(\mathbf{R}_1\cap\mathbf{R}_2)\cup(\mathbf{W}_1\cap\mathbf{R}_2)]\}\cup\{\mathbf{U}_B\cap[(\mathbf{R}_1\cap\mathbf{R}_2)\cup(\mathbf{W}_1\cap\mathbf{R}_2)]\} =$

$\{(\mathbf{U}_A\cap\mathbf{R}_1\cap\mathbf{R}_2)\cup(\mathbf{U}_A\cap\mathbf{W}_1\cap\mathbf{R}_2)]\}\cup\{(\mathbf{U}_B\cap\mathbf{R}_1\cap\mathbf{R}_2)\cup(\mathbf{U}_B\cap\mathbf{W}_1\cap\mathbf{R}_2)\}$

$P(\mathbf{R}_2) = P(\mathbf{U}_A\cap\mathbf{R}_1\cap\mathbf{R}_2) + P(\mathbf{U}_A\cap\mathbf{W}_1\cap\mathbf{R}_2) + P(\mathbf{U}_B\cap\mathbf{R}_1\cap\mathbf{R}_2) + P(\mathbf{U}_B\cap\mathbf{W}_1\cap\mathbf{R}_2)\}) =$

$P(\mathbf{U}_A)P(\mathbf{R}_1|\mathbf{U}_A)P(\mathbf{R}_2|\mathbf{U}_A\cap\mathbf{R}_1) + P(\mathbf{U}_A)P(\mathbf{W}_1|\mathbf{U}_A)P(\mathbf{R}_2|\mathbf{U}_A\cap\mathbf{W}_1) +$

$P(\mathbf{U}_B)P(\mathbf{R}_1|\mathbf{U}_B)P(\mathbf{R}_2|\mathbf{U}_B\cap\mathbf{R}_1) + P(\mathbf{U}_B)P(\mathbf{W}_1|\mathbf{U}_B)P(\mathbf{R}_2|\mathbf{U}_B\cap\mathbf{W}_1) =$

$(1/4)(6/9)(9/20) + (1/4)(3/9)(8/20) + (3/4)(8/19)(7/10) + (3/4)(11/19)(6/10) =$

$54/720 + 24/720 + 168/760 + 198/760 = 78/720 + 366/760 = 13/120 + 183/380 = 269/456$

48.
$\mathbf{R}_1$: The first marble selected is red.
$\mathbf{B}_1$: The first marble selected is black..
$\mathbf{R}_2$: The second marble selected is red.
$\mathbf{B}_2$: The second marble selected is black.
H: The coin tossed resulted in heads.

T: The coin tossed resulted in tails.

➤ a.
$\mathbf{R}_1 = (\mathbf{R}_1 \cap \mathbf{H}) \cup (\mathbf{R}_1 \cap \mathbf{T})$

$P(\mathbf{R}_1) = P(\mathbf{R}_1 \cap \mathbf{H}) + P(\mathbf{R}_1 \cap \mathbf{T}) = P(\mathbf{H})P(\mathbf{R}_1|\mathbf{H}) + P(\mathbf{T})P(\mathbf{R}_1|\mathbf{T}) = (1/2)(5/10) + (1/2)(3/10) = 8/20 = 2/5.$

$P(\mathbf{B}_2) = P(\mathbf{B}_2 \cap \mathbf{H}) + P(\mathbf{B}_2 \cap \mathbf{T}) = P(\mathbf{H})P(\mathbf{B}_2|\mathbf{H}) + P(\mathbf{T})P(\mathbf{B}_2|\mathbf{T}) = (1/2)(7/10) + (1/2)(5/10) = 12/20 = 3/5$

$P(\mathbf{R}_1)P(\mathbf{B}_2) = (2/5)(3/5) = 6/25$

$\mathbf{R}_1 \cap \mathbf{B}_2 = [(\mathbf{R}_1 \cap \mathbf{B}_2) \cap \mathbf{H}] \cup [(\mathbf{R}_1 \cap \mathbf{B}_2) \cap \mathbf{T}]$

$P(\mathbf{R}_1 \cap \mathbf{B}_2) = P[(\mathbf{R}_1 \cap \mathbf{B}_2) \cap \mathbf{H}] + P[(\mathbf{R}_1 \cap \mathbf{B}_2) \cap \mathbf{T}] = P(\mathbf{H})P[(\mathbf{R}_1 \cap \mathbf{B}_2)|\mathbf{H}] + P(\mathbf{T})P[(\mathbf{R}_1 \cap \mathbf{B}_2)|\mathbf{T}] =$

$(1/2)(5/10)(7/10) + (1/2)(3/10)(5/10) = 35/200 + 15/200 = 50/200 = 1/4$

Therefore, $P(\mathbf{R}_1 \cap \mathbf{B}_2) \neq P(\mathbf{R}_1)P(\mathbf{B}_2)$ and the events $\mathbf{R}_1$ and $\mathbf{B}_2$ are not independent events.

➤b.
$P(\mathbf{R}_1 \mid \mathbf{H}) = 5/10$

$P(\mathbf{B}_2 \mid \mathbf{H}) = 7/10$

$P(\mathbf{R}_1 \cap \mathbf{B}_2 \mid \mathbf{H}) = (5/10)(7/10) = P(\mathbf{R}_1 \mid \mathbf{H})P(\mathbf{B}_2 \mid \mathbf{H})$

Therefore, $\mathbf{R}_1$ and $\mathbf{B}_2$ are conditionally independent relative to **H**.

49.
➤ a.
Step 1: Since the die is tossed twice the sample space of the sum of the 2 possible number is $\#\mathbf{S} = 36$.

Step 2: **A**: (2,1), (2,2), (2,3), (2,4), (2,5), (2,6). $P(\mathbf{A}) = 6/36 = 1/6$.

B: (1,4), (2,4), (3,4), (4,4), (5,4), (6,4). $P(\mathbf{B}) = 6/36 = 1/6$

$\mathbf{A} \cap \mathbf{B}$: (2,4). $P(\mathbf{A} \cap \mathbf{B}) = 1/36 = P(\mathbf{A})P(\mathbf{B}) = (1/6)(1/6)$

➤ b.
Step 1: **C**: (1,5),(2,4),(3,3),(4,2),(5,1)

$\mathbf{A} \cap \mathbf{C} = (2,4)$

Step 2: P(**A** | **C**) = P(**A**∩**C**)/P(**C**).

P(**A**∩**C**) = 1/36

P**C**) -= 5/36

P(**A**) = 1/6

P(**A** |**C**) = P(**A**∩**C**)/P(**C**) = (1/36)(36/5) = 1/5 ≠ P(**A**) = 1/6

Therefore the events **A** and **B** are not independent and there the three events are not mutually independent.

➤ c.
P(**A** | **C**) = 1/5

P(**B** | **C**) = 1/5

P(**A** | **C**)P(**B** | **C**) = (1/5)(1/5) = 1/25

P(**A**∩**B** | **C**) = 1/5

Since P(**A**∩**B** | **C**) ≠ P(**A** | **C**)P(**B** | **C**), the events **A** and **B** are not conditionally independent.

50.
➤ a.
S = {(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6),
(3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6),
(5,1), (5,2), (5,3), (5,4), (5,5), (,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)}

$\mathbf{T}_1$ = {(2,1), (2,2), (2,3), (2,4), (2,5), (2,6)}
$\mathbf{T}_2$ = {(1,2),(2,2),(3,2),(4,2),(5,2),(6,2)}

#**S** = 36

#$\mathbf{T}_1$ = 6
#$\mathbf{T}_2$ = 6
#($\mathbf{T}_1 \cap \mathbf{T}_2$) = 1

Therefore, P($\mathbf{T}_1$) = 6/36 = 1/6, P($\mathbf{T}_2$) = 6/36 = 1/6, P(($\mathbf{T}_1 \cap \mathbf{T}_2$) = 1/36.

This shows independence since P($\mathbf{T}_1 \cap \mathbf{T}_2$) = 1/36 = (1/6)(1/6) = P($\mathbf{T}_1$) P($\mathbf{T}_2$) .

➤ b.

$\mathbf{E}_1 \cap \mathbf{E}_2 = \{(2,2)(2,4)(2,6)\ (4,2)(4,4)(4,6)\ (6,2)(6,4)(6,6)\}$

$\#(\mathbf{E}_1 \cap \mathbf{E}_2) = 9$.

Step 1. $P(\mathbf{T}_1 \cap \mathbf{T}_2 \mid \mathbf{E}_1 \cap \mathbf{E}_2) = 1/9$

Step 2: $P(\mathbf{T}_1 \mid \mathbf{E}_1 \cap \mathbf{E}_2) = 3/9 = 1/3$

Step 3: $P(\mathbf{T}_2 \mid \mathbf{E}_1 \cap \mathbf{E}_2) = 3/9 = 1/3$

Step 4: Therefore, $P(\mathbf{T}_1 \cap \mathbf{T}_2 \mid \mathbf{E}_1 \cap \mathbf{E}_2) = 1/9 = (1/3)(1/3) = P(\mathbf{T}_1 \mid \mathbf{E}_1 \cap \mathbf{E}_2) P(\mathbf{T}_2 \mid \mathbf{E}_1 \cap \mathbf{E}_2)$

51.

Step 1: $P(\mathbf{A} \cap \mathbf{B} \mid \mathbf{C} \cap \mathbf{D}) = P(\mathbf{A} \cap \mathbf{B} \cap \mathbf{C} \cap \mathbf{D}) / P(\mathbf{C} \cap \mathbf{D})$

Step 2: Since $\mathbf{A} \subseteq \mathbf{C}$ and $\mathbf{B} \subseteq \mathbf{D}$ we have $\mathbf{A} \cap \mathbf{B} \cap \mathbf{C} \cap \mathbf{D} = \mathbf{A} \cap \mathbf{B}$

Step 3: $P(\mathbf{A} \cap \mathbf{B} \mid \mathbf{C} \cap \mathbf{D}) = P(\mathbf{A} \cap \mathbf{B}) / P(\mathbf{C} \cap \mathbf{D})$

Step 4: Since the pair **A**, **B** is independent and **C**,**D** independent, we have

$P(\mathbf{A} \cap \mathbf{B} \mid \mathbf{C} \cap \mathbf{D}) = P(\mathbf{A})P(\mathbf{B}) / P(\mathbf{C})P(\mathbf{D}) = [P(\mathbf{A})/P(\mathbf{C})][P(\mathbf{B})/P(\mathbf{D})]$

Step 5: $\mathbf{A} = \mathbf{A} \cap \mathbf{C}$ and $\mathbf{B} = \mathbf{B} \cap \mathbf{D}$

Step 6: $P(\mathbf{A} \cap \mathbf{B} \mid \mathbf{C} \cap \mathbf{D}) = [P(\mathbf{A})/P(\mathbf{C})][P(\mathbf{B})/P(\mathbf{D})] = P(\mathbf{A} \cap \mathbf{C})/P(\mathbf{C})][P(\mathbf{B} \cap \mathbf{D})/P(\mathbf{D})] =$

$P(\mathbf{A} \mid \mathbf{C}) P(\mathbf{B} \mid \mathbf{D})$

52.

➤a.

Step 1: $P(\mathbf{A} \cap \mathbf{B} \mid \mathbf{C}) + P(\mathbf{A}' \cap \mathbf{B} \mid \mathbf{C}) = P[(\mathbf{A} \cap \mathbf{B}) \cup (\mathbf{A}' \cap \mathbf{B}) \mid \mathbf{C}) = P(\mathbf{B} \mid \mathbf{C})$

Step 2: $P(\mathbf{A} \mid \mathbf{C}) P(\mathbf{B} \mid \mathbf{C}) + P(\mathbf{A}' \cap \mathbf{B} \mid \mathbf{C}) = P(\mathbf{B} \mid \mathbf{C})$

Step 3: $P(\mathbf{A}' \cap \mathbf{B} \mid \mathbf{C}) = P(\mathbf{B} \mid \mathbf{C}) - P(\mathbf{A} \mid \mathbf{C}) P(\mathbf{B} \mid \mathbf{C}) = P(\mathbf{B} \mid \mathbf{C})[1 - P(\mathbf{A} \mid \mathbf{C})] = P(\mathbf{A}' \mid \mathbf{C}) P(\mathbf{B} \mid \mathbf{C})$

➤ b.

From a. we have **A**′ and **B** are conditionally independent relative to **C**

Applying a. to the conditionally independent pair **A**′ and **B**, it follows pair **A**′ and **B**′ is also conditionally independent relative to **C**.

53. The sample space generated by tossing a fair die 3 time has cardinality #**S =** (6)(6)(6) = 6^3.

Step 1: $\mathbf{A}_{1,2}$ = {(1,1,x1),(2,2,x2), (3,3,x3), (4,4,x4), (5,5,x5), (6,6,x6)}, where x1, ... , x6 = 1,2,3,4,5,6.

#$\mathbf{A}_{1,2}$ = (6)(6) = 6^2

P($\mathbf{A}_{1,2}$) = $6^2/6^3$ = 1/6

$\mathbf{A}_{1,3}$ = {(1,x1,1),(2,x2,2), (3,x3,3), (4,x4,4), (5,x5,5), (6,x6,6)}, where x1, ..,. x6 = 1,2,3,4,5,6.

#$\mathbf{A}_{1,3}$ = (6)(6) = 36

P($\mathbf{A}_{1,3}$) = $6^2/6^3$ = 1/6

$\mathbf{A}_{2,3}$ = {(x1,1,1),(x2,2,2), (x3,3,3), (x4,4,4), (x5,5,5), (x6,6,6)}, where x1, ..., x6 = 1,2,3,4,5,6.

#$\mathbf{A}_{2,3}$ = (6)(6) = 36

P($\mathbf{A}_{2,3}$) = $6^2/6^3$ = 1/6

Step 2: $\mathbf{A}_{1,2}\cap\mathbf{A}_{1,3}$ = (1,1,1),(2,2,2), (3,3,3), (4,4,4), (5,5,5), (6,6,6)}

P($\mathbf{A}_{1,2}\cap\mathbf{A}_{1,3}$) = $6/6^3$ = $1/6^2$ = (1/6)(1/6) = P($\mathbf{A}_{1,2}$)P($\mathbf{A}_{1,3}$)

$\mathbf{A}_{1,2}\cap\mathbf{A}_{2,3}$ = (1,1,1),(2,2,2), (3,3,3), (4,4,4), (5,5,5), (6,6,6)}

P($\mathbf{A}_{1,2}\cap\mathbf{A}_{2,3}$) = $6/6^3$ = $1/6^2$ = (1/6)(1/6) = P($\mathbf{A}_{1,2}$)P($\mathbf{A}_{2,3}$)

$\mathbf{A}_{1,3}\cap\mathbf{A}_{2,3}$ = {(1,1,1),(2,2,2), (3,3,3), (4,4,4), (5,5,5), (6,6,6)}

P($\mathbf{A}_{1,3}\cap\mathbf{A}_{2,3}$) = $6/6^3$ = $1/6^2$ = (1/6)(1/6) = P($\mathbf{A}_{1,3}$)P($\mathbf{A}_{2,3}$)

Step 3: $\mathbf{A}_{1,2}\cap\mathbf{A}_{1,3}\cap\mathbf{A}_{2,3}$ = {(1,1,1),(2,2,2), (3,3,3), (4,4,4), (5,5,5), (6,6,6)}

P($\mathbf{A}_{1,2}\cap\mathbf{A}_{1,3}\cap\mathbf{A}_{2,3}$) = $6/6^3$ = $1/6^2$

P($\mathbf{A}_{1,2}$)P($\mathbf{A}_{1,3}$)P($\mathbf{A}_{2,3}$) = $1/6^3$

Since P($\mathbf{A}_{1,2}\cap\mathbf{A}_{1,3}\cap\mathbf{A}_{2,3}$) $\neq$ P($\mathbf{A}_{1,2}$)P($\mathbf{A}_{1,3}$)P($\mathbf{A}_{2,3}$)

The three events are not mutually independent.

Since the probability of each pair is equal the three events are par wise independent.

54.
$\mathbf{F}_1$: First card drawn is a face card.
$\mathbf{D}_2$: Second card drawn is a diamond.
$\mathbf{K}_3$: Third card drawn is a king.

$\mathbf{E} = \mathbf{F}_1 \cap \mathbf{D}_2 \cap \mathbf{K}_3 =$

$[(\mathbf{F}_1 \cap \mathbf{D}_1 \cap \mathbf{K}_1) \cap (\mathbf{D}_2 \cap \mathbf{K}_2) \cap \mathbf{K}_3] \cup [(\mathbf{F}_1 \cap \mathbf{D}_1 \cap \mathbf{K}_1') \cap (\mathbf{D}_2 \cap \mathbf{K}_2) \cap \mathbf{K}_3] \cup$

$[(\mathbf{F}_1 \cap \mathbf{D}_1' \cap \mathbf{K}_1) \cap (\mathbf{D}_2 \cap \mathbf{K}_2) \cap \mathbf{K}_3] \cup [(\mathbf{F}_1 \cap \mathbf{D}_1' \cap \mathbf{K}_1') \cap (\mathbf{D}_2 \cap \mathbf{K}_2) \cap \mathbf{K}_3] \cup$

$[(\mathbf{F}_1 \cap \mathbf{D}_1 \cap \mathbf{K}_1) \cap (\mathbf{D}_2 \cap \mathbf{K}_2') \cap \mathbf{K}_3] \cup [(\mathbf{F}_1 \cap \mathbf{D}_1 \cap \mathbf{K}_1') \cap (\mathbf{D}_2 \cap \mathbf{K}_2') \cap \mathbf{K}_3] \cup$

$[(\mathbf{F}_1 \cap \mathbf{D}_1' \cap \mathbf{K}_1) \cap (\mathbf{D}_2 \cap \mathbf{K}_2') \cap \mathbf{K}_3] \cup [(\mathbf{F}_1 \cap \mathbf{D}_1' \cap \mathbf{K}_1') \cap (\mathbf{D}_2 \cap \mathbf{K}_2') \cap \mathbf{K}_3] =$

$P[(\mathbf{F}_1 \cap \mathbf{D}_1 \cap \mathbf{K}_1) \cap (\mathbf{D}_2 \cap \mathbf{K}_2) \cap \mathbf{K}_3] + P[(\mathbf{F}_1 \cap \mathbf{D}_1 \cap \mathbf{K}_1') \cap (\mathbf{D}_2 \cap \mathbf{K}_2) \cap \mathbf{K}_3] +$

$P[(\mathbf{F}_1 \cap \mathbf{D}_1' \cap \mathbf{K}_1) \cap (\mathbf{D}_2 \cap \mathbf{K}_2) \cap \mathbf{K}_3] + P[(\mathbf{F}_1 \cap \mathbf{D}_1' \cap \mathbf{K}_1') \cap (\mathbf{D}_2 \cap \mathbf{K}_2) \cap \mathbf{K}_3] +$

$P[(\mathbf{F}_1 \cap \mathbf{D}_1 \cap \mathbf{K}_1) \cap (\mathbf{D}_2 \cap \mathbf{K}_2') \cap \mathbf{K}_3] + P[(\mathbf{F}_1 \cap \mathbf{D}_1 \cap \mathbf{K}_1') \cap (\mathbf{D}_2 \cap \mathbf{K}_2') \cap \mathbf{K}_3] +$

$P[(\mathbf{F}_1 \cap \mathbf{D}_1' \cap \mathbf{K}_1) \cap (\mathbf{D}_2 \cap \mathbf{K}_2') \cap \mathbf{K}_3] + P[(\mathbf{F}_1 \cap \mathbf{D}_1' \cap \mathbf{K}_1') \cap (\mathbf{D}_2 \cap \mathbf{K}_2') \cap \mathbf{K}_3] =$

(1/52)(0) + (2/52)(1/51)(3/50) + (3/52)(1/51)(2/50) + (6/52)(1/51)(3/50) +

(1/52)(12/51)(3/50) + (2/52)(11/52)(4/52) + (3/52)(12/51)(3/50) + (6/52)(12/52)(4/52)

$$(\frac{1}{52})(0) + (\frac{2}{52})(\frac{1}{51})(\frac{3}{50}) + (\frac{3}{52})(\frac{1}{51})(\frac{2}{50}) + (\frac{6}{52})(\frac{1}{51})(\frac{3}{50}) +$$
$$(\frac{1}{52})(\frac{12}{51})(\frac{3}{50}) + (\frac{2}{52})(\frac{11}{51})(\frac{4}{50}) + (\frac{3}{52})(\frac{12}{51})(\frac{3}{50}) + (\frac{6}{52})(\frac{12}{51})(\frac{4}{50}) = \frac{550}{132600}$$

55.
$\mathbf{R}_1$: The first marble selected is red.
$\mathbf{B}_2$: The second marble selected is black
$\mathbf{U}_1$: Urn 1 is selected.
$\mathbf{U}_2$: Urn 2 is selected.
$\mathbf{U}_3$: Urn 3 is selected

$P(\mathbf{B}_2 \mid \mathbf{R}_1) = P(\mathbf{R}_1 \cap \mathbf{B}_2)/\ P(\mathbf{R}_1)$

Step 1: $\mathbf{R}_1 = (\mathbf{R}_1 \cap \mathbf{U}_1) \cup (\mathbf{R}_1 \cap \mathbf{U}_2) \cup (\mathbf{R}_1 \cap \mathbf{U}_3)$

$P(\mathbf{R}_1) = P(\mathbf{R}_1 \cap \mathbf{U}_1) + P(\mathbf{R}_1 \cap \mathbf{U}_2) + P(\mathbf{R}_1 \cap \mathbf{U}_3) = P(\mathbf{U}_1)P(\mathbf{R}_1|\mathbf{U}_1) + P(\mathbf{U}_2)P(\mathbf{R}_1|\mathbf{U}_2) + P(\mathbf{U}_1)P(\mathbf{R}_1|\mathbf{U}_2) =$

$(1/3)(3/10) + (1/3)(12/20) + (1/3)(5/10) = (1/3)(28/20) = 28/60 = 7/15$

Step 2: $\mathbf{R}_1 \cap \mathbf{B}_2 = [(\mathbf{R}_1 \cap \mathbf{B}_2) \cap \mathbf{U}_1] \cup [(\mathbf{R}_1 \cap \mathbf{B}_2) \cap \mathbf{U}_2] \cup [(\mathbf{R}_1 \cap \mathbf{B}_2) \cap \mathbf{U}_3]$

$P(\mathbf{R}_1 \cap \mathbf{B}_2) = P\{[(\mathbf{R}_1 \cap \mathbf{B}_2) \cap \mathbf{U}_1)] \cup [(\mathbf{R}_1 \cap \mathbf{B}_2) \cap \mathbf{U}_2\ ] \cup [(\mathbf{R}_1 \cap \mathbf{B}_2) \cap \mathbf{U}_3)]\} =$

$P[(\mathbf{R}_1 \cap \mathbf{B}_2) \cap \mathbf{U}_1)] + P[(\mathbf{R}_1 \cap \mathbf{B}_2) \cap \mathbf{U}_2\ ] + P[(\mathbf{R}_1 \cap \mathbf{B}_2) \cap \mathbf{U}_3)]\} =$

$P(\mathbf{U}_1)P(\mathbf{R}_1|\mathbf{U}_1)P(\mathbf{R}_1 \cap \mathbf{B}_2|\mathbf{U}_1 \cap \mathbf{R}_1) + P(\mathbf{U}_2)P(\mathbf{R}_1|\mathbf{U}_2)P(\mathbf{R}_1 \cap \mathbf{B}_2|\mathbf{U}_2 \cap \mathbf{R}_1) +$

$P(\mathbf{U}_3)P(\mathbf{R}_1|\mathbf{U}_3)P(\mathbf{R}_1 \cap \mathbf{B}_2|\mathbf{U}_3 \cap \mathbf{R}_1) = (1/3)(3/10)(7/9) + (1/3)(12/20)(8/19) + (1/3)(5/10)(5/9) =$

$$\frac{653}{2565}$$

Step 3: $P(\mathbf{B}_2 \mid \mathbf{R}_1) = P(\mathbf{R}_1 \cap \mathbf{B}_2)/\ P(\mathbf{R}_1) = \dfrac{\frac{653}{2565}}{\frac{7}{15}} = \left(\dfrac{653}{2665}\right)\left(\dfrac{15}{7}\right) = \dfrac{1959}{3731}$

56.

$\mathbf{T}_1$: Table 1 is selected.
$\mathbf{T}_2$: Table 2 is selected.
$\mathbf{U}_1$: Urn 1 sitting on Table 1 is selected.
$\mathbf{U}_2$: Urn 2 sitting on Table 1 is selected.
$\mathbf{U}_3$: Urn 3 sitting on Table 2 is selected.
$\mathbf{U}_4$: Urn 4 sitting on Table 2 is selected.
$\mathbf{R}$: Red marble is selected.

TABLE 1		**TABLE 2**	
URN 1	**URN 2**	**URN 3**	**URN 4**
12 red marbles 8 white marbles	15 red marbles 5 white marbles	10 red marbles 10 white marbles	5 red marbles 15 red marbles

A table and an urn sitting on the tab

$\mathbf{R} = (\mathbf{R}\cap\mathbf{T}_1\cap\mathbf{U}_1)\cup(\mathbf{R}\cap\mathbf{T}_1\cap\mathbf{U}_2)\cup(\mathbf{R}\cap\mathbf{T}_2\cap\mathbf{U}_3)\cup(\mathbf{R}\cap\mathbf{T}_2\cap\mathbf{U}_4)$

$P(\mathbf{R}) = P(\mathbf{R}\cap\mathbf{T}_1\cap\mathbf{U}_1) + P(\mathbf{R}\cap\mathbf{T}_1\cap\mathbf{U}_2) + P(\mathbf{R}\cap\mathbf{T}_2\cap\mathbf{U}_3) + P(\mathbf{R}\cap\mathbf{T}_2\cap\mathbf{U}_4) =$

$P(\mathbf{T}_1)P(\mathbf{U}_1|\mathbf{T}_1)P(\mathbf{R}|\mathbf{T}_1\cap\mathbf{U}_1) + P(\mathbf{T}_1)P(\mathbf{U}_2|\mathbf{T}_1)P(\mathbf{R}|\mathbf{T}_1\cap\mathbf{U}_2) + P(\mathbf{T}_2)P(\mathbf{U}_3|\mathbf{T}_2)P(\mathbf{R}|\mathbf{T}_2\cap\mathbf{U}_3) +$

$P(\mathbf{T}_2)P(\mathbf{U}_4|\mathbf{T}_2)P(\mathbf{R}|\mathbf{T}_2\cap\mathbf{U}_4) = (1/2)(1/2)(12/20) + (1/2)(1/2)(15/20) + (1/2)(1/2)(10/20) +$

$(1/2)(1/2)(5/20) = 12/80 + 15/80 + 10/80 + 5/80 = 42/80$

57.
A: Urn A is selected.
B: Urn B is selected.
C: Urn C is selected.

$\mathbf{R}_a$: Red marble is selected from urn A.
$\mathbf{R}_b$: Red marble is selected from urn B.
$\mathbf{R}_c$: Red marble is selected from urn C.
$\mathbf{R}_1$: First marble selected is red.
$\mathbf{R}_2$: Second marble selected is red.

Step 1: $\mathbf{R}_1 = [(\mathbf{A}\cap\mathbf{B})\cap\mathbf{R}_1]\cup[(\mathbf{A}\cap\mathbf{C})\cap\mathbf{R}_1]\cup[(\mathbf{B}\cap\mathbf{C})\cap\mathbf{R}_1]$

$P(\mathbf{R}_1) = P[(\mathbf{A}\cap\mathbf{B})\cap\mathbf{R}_1] + P[(\mathbf{A}\cap\mathbf{C})\cap\mathbf{R}_1] + P[(\mathbf{B}\cap\mathbf{C})\cap\mathbf{R}_1] =$

$P(\mathbf{A}\cap\mathbf{B})P(\mathbf{R}_1|\mathbf{A}\cap\mathbf{B}) + P(\mathbf{A}\cap\mathbf{C})P(\mathbf{R}_1|\mathbf{A}\cap\mathbf{C}) + P(\mathbf{B}\cap\mathbf{C})P(\mathbf{R}_1|\mathbf{B}\cap\mathbf{C}) =$

$(1/3)P(\mathbf{R}_a) + (1/3)P(\mathbf{R}_a) + (1/3)P(\mathbf{R}_b) = (1/3)(3/10 + 3/10 + 7/10) = (1/3)(13/10) = 13/30$

$\mathbf{R}_2 = [(\mathbf{A}\cap\mathbf{B})\cap\mathbf{R}_2]\cup[(\mathbf{A}\cap\mathbf{C})\cap\mathbf{R}_2]\cup[(\mathbf{B}\cap\mathbf{C})\cap\mathbf{R}_2]$

$P(\mathbf{R}_2) = P[(\mathbf{A}\cap\mathbf{B})\cap\mathbf{R}_2] + P[(\mathbf{A}\cap\mathbf{C})\cap\mathbf{R}_2] + P[(\mathbf{B}\cap\mathbf{C})\cap\mathbf{R}_2] =$

$P(\mathbf{A}\cap\mathbf{B})P(\mathbf{R}_2|\mathbf{A}\cap\mathbf{B}) + P(\mathbf{A}\cap\mathbf{C})P(\mathbf{R}_2|\mathbf{A}\cap\mathbf{C}) + P(\mathbf{B}\cap\mathbf{C})P(\mathbf{R}_2|\mathbf{B}\cap\mathbf{C}) =$

$(1/3)P(\mathbf{R}_b) + (1/3)P(\mathbf{R}_c) + (1/3)P(\mathbf{R}_c) = (1/3)(7/10 + 5/10 + 5/10) = (1/3)(17/10) = 17/30$

Step 2: $\mathbf{R}_1\cap\mathbf{R}_2 = [(\mathbf{A}\cap\mathbf{B})\cap(\mathbf{R}_1\cap\mathbf{R}_2)]\cup[(\mathbf{A}\cap\mathbf{C})\cap(\mathbf{R}_1\cap\mathbf{R}_2)]\cup[(\mathbf{B}\cap\mathbf{C})\cap(\mathbf{R}_1\cap\mathbf{R}_2)]$

$P(\mathbf{R}_1\cap\mathbf{R}_2) = P[(\mathbf{A}\cap\mathbf{B})\cap(\mathbf{R}_1\cap\mathbf{R}_2)] + P[(\mathbf{A}\cap\mathbf{C})\cap(\mathbf{R}_1\cap\mathbf{R}_2)] + P[(\mathbf{B}\cap\mathbf{C})\cap(\mathbf{R}_1\cap\mathbf{R}_2)] =$

$P[(\mathbf{A}\cap\mathbf{B})P(\mathbf{R}_1\cap\mathbf{R}_2|\mathbf{A}\cap\mathbf{B}) + P[(\mathbf{A}\cap\mathbf{C})P(\mathbf{R}_1\cap\mathbf{R}_2|\mathbf{A}\cap\mathbf{C}) + P[(\mathbf{B}\cap\mathbf{C})P(\mathbf{R}_1\cap\mathbf{R}_2|\mathbf{B}\cap\mathbf{C}) =$

$(1/3)P(\mathbf{R}_a)P(\mathbf{R}_b) + (1/3)P(\mathbf{R}_a)P(\mathbf{R}_c) + (1/3)P(\mathbf{R}_b)P(\mathbf{R}_c) =$

$(1/3)[(3/10)(7/10) + (3/10)(5/10) + (7/10)(5/10)] = (1/3)[21/100 + 15/100 + 35/100] = 71/300$

Step 3: For independence of $\mathbf{R}_1$ and $\mathbf{R}_2$, we need $P(\mathbf{R}_1 \cap \mathbf{R}_2) = P(\mathbf{R}_1)P(\mathbf{R}_2)$.

$P(\mathbf{R}_1)P(\mathbf{R}_2) = (13/30)(17/30) = 221/900$

$P(\mathbf{R}_1 \cap \mathbf{R}_2) = 71/300 = 213/900$

Since $P(\mathbf{R}_1 \cap \mathbf{R}_2) \neq P(\mathbf{R}_1)P(\mathbf{R}_2)$, we conclude the events $\mathbf{R}_1$ and $\mathbf{R}_2$ are not independent.

58.
R: A red marble is selected.
A: Urn A is selected.
B: Urn B is selected.

$P(\mathbf{A}|\mathbf{R}) = P(\mathbf{A} \cap \mathbf{R})/P(\mathbf{R})$
$P(\mathbf{A} \cap \mathbf{R}) = P(\mathbf{A})P(\mathbf{R}|\mathbf{A}) = (1/6)(3/8)$
$\mathbf{R} = (\mathbf{R} \cap \mathbf{A}) \cup (\mathbf{R} \cap \mathbf{B})$
$P(\mathbf{R}) = P(\mathbf{R} \cap \mathbf{A}) + P(\mathbf{R} \cap \mathbf{B}) = P(\mathbf{A})P(\mathbf{R}|\mathbf{A}) + P(\mathbf{B})P(\mathbf{R}|\mathbf{B}) = (1/6)(3/8)+(5/6)(7/10) = 31/48$