PROBABILITY THEORY
Lesson 11
Sampling with and Without Replacement

11.1- Counting Tasks

11.1 - Problem 1:
Step 1: From the first urn, there are n = 10 possible selections.

Step 2: From the second urn, there are m = 15 possible selections.

Step 3:Therefore, the total number of possible selections is nxm = 10x15 = 150.

11.1 - Problem 2:
Step 1: She has one of 12 hats to choose from: n = 12

Step 2: She has one of 15 pairs of shoes to choose from: m = 15

Step 3: She has one of 20 dresses to choose from: r = 20

Therefore, the total number of possible selections is nxmxr = 12x15x20 = 3,600.

11.2 -What is Sampling With Replacement?

11.2 - Problem 1:
➤(a).
{1, 2, 3, 4, 5, 6}

➤(b).
Each toss of the die can result in one of 6 possible numbers. Therefore, r = 2 and n = 6 and #**S** = 6^2 = 36.

➤(c).
S = {(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6),
(3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6),
(5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)}

The first number in each pair represents the number appearing on the first toss and the second number in each pair represents the number appearing on the second toss.

➤(d).
#**S** = 36, which is the number of pairs of numbers in the sample space **S**.

Let **E** be the event that all the tosses resulted in even numbers. Therefore,

E = {(2,2), (2,4), (2,6), (4,2), (4,4), (4,6), (6,2), (6,4), (6,6)}

P(**E**) = 9/ 36

➤(e).
Let **E** be the event that all the tosses resulted in the same numbers. Therefore,

E = {(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)}.

P(**E**) = 6/36

➤(f).
Let **E** be the event that the sum of the two numbers equal 5. Therefore,

E = {(1,4), (2,3), (3,2), (4,1)}.

Therefore, P(**E**) = 4/36.

11.2 - Problem 2:
Step 1: u: a driver is driving under the influence.

s: a driver is speeding.

Step 2: Since 6 drivers were selected, there are two possible selections on the first, two on the second, etc.

Therefore, #**S** = 2x2x2x2x2x2 = 64.

Step 3: **E** = *five drivers were stopped for speeding and one for driving under the influence.*

= {(u,s,s,s,s,s),(s,u,s,s,s,s),(s,s,u,s,s,s),(s,s,s,u,s,s),(s,s,s,s,u,s),(s,s,s,s,s,u)}

Step 4: P(**E**) = #**E**/#**S** = 6/64.

11.3 - What is Sampling Without Replacement?

11.3 - Problem 1:
➤(a).
Since the urn contains three marble marked 1,2, and 3 respectively. Therefore, the original sample space is {1,2,3).

➤(b).
Step 1: Since the sampling is without replacement,

S = {(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)}

Here, for example (2,3,1) means that the marble marked 2 was selected first, the marble marked 3 was selected second and the marlbe marked 1 was selected third.

➤(c).
Since **S** = {(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)},

#**S** = 6.

11.3 - Problem 2:
➤(a).
Assume the first selection is prize one, second selection is prize two and third selection is prize three.

The first selection has 5 + 7 = 12 possible choices.

The second selections has 11 possible choices

The third selection has 10 possible choices.

Therefore, #**S** = 12x11x10 = 1,320.

➤(b).
There are 7 girls.

The first selection has 7 possible choices.
The second selection has 6 possible choices.
The third selection has 5 possible choices.

E = *that all three students receiving prizes were girls*

#**E** = 7x6x5 = 210

Therefore, P(**E**) = 210/1320.

Supplementary Problems

1.
Step 1: For the sample space, we use the following symbols:

u: stopped for driving under the influencing

s: stopped for speeding

Step 2: For each driver stopped, there are two possibilities. Therefore, the cardinality of the sample space is

#**S** = 2x2x2x2x2x2 = 64.

Step 3: **E** = {(s,s,s,s,u,u),(s,s,s,u,s,u),(s,s,u,s,s,u),(s,u,s,s,s,u),(u,s,s,s,s,u),

(s,s,s,u,u,s),(s,s,u,s,u,s),(s,u,s,s,u,s),(u,s,s,s,u,s),(s,s,u,u,s,s),

(s,u,s,u,s,s),(u,s,s,u,s,s),(s,u,u,s,s,s),(u,s,u,s,s,s),(u,u,s,s,s,s)}

Step 4: P(**E**) = 15/64

2.
Step 1: For each position on the license plates there are a total of 10 + 26 = 36 digits and letters.

Step 2: For each of the 5 characters on the plate there are 36 possible selections.

Therefore,

#**S** = 36x36x36x36 = 60,466,176

Step 3: The number of ways the first two characters are digits is n = 10x10 = 100.

Step 4: The number of ways the last three characters are letters is m = 26x26x26 = 17,576.

Step 5: To count the number of ways the first two charters are digits and the last three characters are letters is nxm = 100x17,576 = 1,757,600. Therefore,

#**E** = 1,756,600.

Step 6: P(**E**) = #**E**/#**S** = $\frac{1,757,600}{60,466,176}$

3.

Step 1:The original population has 5 + 7 = 12 boys and girls. Since four of them are to receive each a different prize, the number of possible different winners is 12x11x10x9 = 11,880.

Therefore, #**S** = 11,880.

Step 2: The event **E** that two boys and two girls win prizes occurs in six different ways:

1. Two boys win first and second prizes and two girls win third and fourth prizes

(b,b,g,g): 5x4x7x6 = 840.

or

2. Two boy wins first and third prizes and two girls win second and fourth prizes (b,g,b,g):

5x7x4x6 = 840.

or

3. Two boy wins first and fourth prizes and two girls win second and third prizes (b,g,g,b):

5x7x6x4 = 840.

or

4. Two girls win first and second prizes and two boys win third and fourth prizes (g,g,b,b):

7x6x5x4 = 840.

or

5. Two girls wins first and third prizes and two boys win second and fourth prizes (g,b,g,b):

7x5x6x4 = 840.

or

6. Two girls wins first and fourth prizes and two boys win second and third prizes (g,b,b,g):

7x5x4x6 = 840.

Therefore #**E** = 6x840 = 5040.

P(**E**) = $\frac{5040}{11880}$.

4.

➤ a.

Step 1: Since the four cards are taken from the deck without replacement,

#**S** = 52x51x50x49 = 6,497,400

Step 2: **E** = *the hand contains at least one king.*

E′ = *the hand contains no kings.*

Step 3: In an ordinary deck of cards, there are 48 cards that are not kings.

Therefore, #**E**′ = 48x47x46x45 = 4,669,920

Step 4: P(**E**′) = $\frac{4669920}{6497400}$

Step 5: P(**E**) = 1 - P(**E**′) = $1 - \frac{4669920}{6497400} = \frac{1827480}{6497400}$

➤ b.
E = *the event that at most three jacks were selected.*

Step 2: **E**′ = *the event that exactly four jacks were selected.*

Since there are only 4 jacks in the deck, **#E** = 4x3x2x1 = 24

Therefore, P(**E**′) = $\frac{4}{6497376}$.

P(**E**) = $1 - \frac{24}{6497376} = \frac{6497352}{6497376}$.

➤ c.
Step 1: **E** = *the event that all cards drawn are face cards.*

Step 2: There are 12 face cards in an ordinary deck of cards.

Therefore, #**E** = 12x11x10x9 = 11880

Step 3: P(**E**) = $\frac{11880}{6497400}$

➤ d.
Step 1: **E** = *the event that three cards are queens.*

There are 4 queens and 48 non-queens in a ordinary deck of cards.

Step 2: The event **E** can occur in four different ways:

1. queens occur on the first, second and third drawings only:4x3x2x48 = 1152.

or

2. queens occur on the first, second and fourth drawings only: 4x3x48x2 = 1152.

or

3. queens occur on the first, third and fourth drawings only: 4x48x3x2 = 1152.

or

4. queens occur on the second, third and fourth drawings only: 48x4x3x2 = 1152

Step 3: Therefore #**E** = 4x1152 = 4608

Step 4: P(**E**) = $\frac{4608}{6497400}$

5.

Step 1: Assume each student is numbered 1 through N respectively.

Then student 1 can be born on any of the 365 days: $n_1 = 365$.

Then student 2 can be born on any of the 365 days: $n_2 = 365$.

Then student 3 can be born on any of the 365 days: $n_3 = 365$.

:::

Then student N can be born on any of the 365 days: $n_N = 365$.

Therefore, #**S** = $n_1 n_2 n_3 \ldots n_N = 365^N$

Step 2: **E** = *the event that at least two students are born on the same day.*

Step 3: **E**′ = *the event that no students are born on the same day.*

Step 4: The event **E**′ can occur in the following ways:

1. Student 1 can be born on any of the 365 days

and

2. Student 2 can be born on any of the other 364 days

and

3. Student 3 can be born on any of the other 363 days

::

4. Student N can only be born on the remaining 365 - N + 1 days.

5. Therefore, #**E**′ = 365x364x...x(365 - N + 1)

6. Therefore, P(**E**′) = $\frac{(365)(365-1)...(365 - N+1)}{365^N}$

7. Therefore, P(**E**) = $1 - \frac{(365)(365-1)...(365 - N+1)}{365^N}$

6.
Step 1: For each of the three selections, there are three possibilities. Therefore,

#**S** = 3_3 = 27.

Step 2: **R** = all three marbles selected are red: (r,r,r).

#**R** = 1

Step 3: **W** = all three marbles selected are white: (w,w,w).

#**W** = 1

Step 4: **E** = *that all the marbles selected are all red or all white.*

Step 5: **E** = **R** ∪**W**

Step 6: P(**E**) = P(**R**) + P(**W**) = 1/27 + 1/27 = 2/27

7.
➤ a.
Step 1: Since repetition is not allowed, The first selection allows picking any of the ten numbers, the second selection allows picking any of the remaining nine numbers, and the third selection allows picking any of the remaining eight numbers. Therefore,

#**S** = 10x9x8 = 720.

Step 2: #**E** = *the event that numbers 1,2,3 are selected.*

The event **E** can occur in six different ways:

E = {(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)}.

Step 3: Therefore, P(**E**) = 6/720.

➤ b.
Step 1: **E** = *only even numbers are selected.*

Step 2: The even numbers are 2,4,6,8,10.

Step 3: The event **E** occurs in the following ways: one of 5 possible numbers on the first selection, one of 4 possible numbers selected on the second selection, one of 3 possible numbers on the third selection.

Therefore, #**E** = 5x4x3 = 60.

Step 4: P(**E**) = 60/720.

➤ c.
There are 5 even and 5 odd numbers.

Step 1: **E** = *that exactly two even numbers are selected.*

Step 2: The event **E** occurs in the following thee ways:

1. an odd numbers is selected on the first selection and both event numbers occur on the second and third selection: 5x5x4 = 100.

or

2. an odd numbers is selected on the second selection and both event numbers occur on the first and third selection: 5x5x4 = 100.

or

3. an odd numbers is selected on the third selection and both event numbers occur on the first and second selection: 5x4x5 = 100.

Step 3: Therefore, #**E** = 3x100 = 300.

Step 4: P(**E**) = 300/720.

8.
Since the selected each is returned to the deck of cards, there are 52x52x52x52 = 7,311,616 different possible hands.

9.
The population is made up of 10 digits and 26 letters. Therefore, the number of elements in the population is n = 36 and r = 5. Therefore,

36^5 = 60,466,176.

Let **E** = the event that his license plate has 2 numbers and 3 letters. **E** is made up of samples of the form:

(l,l,l,n,n), (l,l,n,n,l), (l,n,n,l,l), (n,n,l,l,l), (l,l,n,l,n), (l,n,l,n,l), (n,l,n,l,l,), (l,n,l,l,n), (n,l,l,n,l),(n,l,l,l,n)}.

But each 5 tuple there is 26^3 x 10^2 = 1,757,600 possibiities. And since there are ten 5 tuples,

#**E** = 10 x 1,757,600 = 17,576,000.

Therefore, P(**E**) = $\frac{17,576,000}{60,466,176}$.

10.
S = {(h,h), (h,t,h), (t,h,h), (t,t,t,t), (h,t,t,t), (t,h,t,t), {t,t,h,t), (t,t,t,h)), (t,t,h,h), {t,h,t,h), (h,t,t,h)}

11.
➤ a.
Since 5 cards are selected without replacement, and there are 52 cards,
#**S** = (52)(51)(50)(49)(48) = 311,875,200.

Since there are 13 clubs in a deck of cards, **#E =** (13)(12)(11)(10)(9)= 154,440

Therefore, P(**E**) = $\frac{154440}{311875200}$.

➤ b.
Let **E** be the event that four cards are clubs. Since four of the cards are clubs and one card is not a club,

#**E** = (5)[(13)(12)(11)(10)(9)][(39)] = 30,115,800

P(**E**) = $\frac{30115800}{311875200}$

12.
➤ a.
Since there are 3 people and 12 months in a year, #**S** = 3^{12} = 531,441.

Let **E** be the event that none were born in the same month.

This can happen in the following way:

The first person can be born in any of the 12 months.

The second person can be born in any of the remaining 11 months.

The third person can be born in any of the remaining 10 months.

Therefore, $\#\mathbf{E} = (12)(11)(10) = 1{,}320$.

$$P(\mathbf{E}) = \frac{1320}{531441}$$

➤ b.
From a., $\mathbf{E}'$ = the event that at least two were born in the same month.

$$P(\mathbf{E}') = 1 - \frac{1320}{531441} = \frac{530121}{531441}$$

13.
Step 1: Define the following events:

$\mathbf{S}_1$: The event a diamond is drawn on the first card.

$\mathbf{S}_2$: The event a diamond is first drawn on the second card.

$\mathbf{S}_3$: The event a diamond is first drawn on the third card.

$\mathbf{S}_4$: The event a diamond is first drawn on the fourth card.

$\mathbf{S}_5$: The event 5 cards are drawn.

$\mathbf{S} = \mathbf{S}_1 \cup \mathbf{S}_2 \cup \mathbf{S}_3 \cup \mathbf{S}_4 \cup \mathbf{S}_5$

Step 2: Since there are 13 diamonds and 39 non-diamonds in an ordinary deck of cards,

$\#\mathbf{S}_1 = 13$

$\#\mathbf{S}_2 = 39(13) = 507$

$\#\mathbf{S}_3 = (39)(38)(13) = 19{,}266$

$\#\mathbf{S}_4 = (39)(38)(37)(13) = 712{,}842$

$\#\mathbf{S}_5 = (39)(38)(37)(36)(13) + (39)(38)(37)(36)(35) = 25{,}662{,}312 + 69{,}090{,}840 = 94{,}753{,}152$

Therefore,

$\#\mathbf{S} = \#\mathbf{S}_1 + \#\mathbf{S}_2 + \#\mathbf{S}_3 + \#\mathbf{S}_4 + \#\mathbf{S}_5 = 13 + 507 + 19{,}266 + 712{,}842 + 94{,}753{,}152 = 95{,}485{,}780$

14.

➤ a.

Since there are 24 shoes in the closet, the number of ways of selecting 6 shoes is

$\#\mathbf{S} = 24\text{x}23\text{x}22\text{x}21\text{x}20\text{x}19 = 96{,}909{,}120$

E: the event that at least 1 pair is selected.

E′: the event that a pair is not selected.

$\#\mathbf{E}' = 24\text{x}22\text{x}20\text{x}18\text{x}16\text{x}14 = 42{,}577{,}920.$

Therefore, $P(\mathbf{E}') = \dfrac{42{,}577{,}920}{96{,}909{,}120}$

and

$$P(\mathbf{E}) = 1 - \frac{42{,}577{,}920}{96{,}909{,}120} = \frac{54{,}331{,}200}{96{,}909{,}120}$$

➤ b.

E: the event that exactly 1 pair is selected.

Since there are 12 pairs of shoes,

$\mathbf{E} = 12\text{x}(22\text{x}20\text{x}18\text{x}16) = 1{,}520{,}640$

$$P(\mathbf{E}) = \frac{1{,}520{,}640}{96{,}909{,}120}$$

15.

➤ a.

The cardinality of the sample space is $\#\mathbf{S} = 36^N$.

Let **E** equal the event that at least one of the pairs occur. For each toss of the pair of dice, the number of ways the above pairs in **A** cannot happen is 6x5 = 30. Therefore,

$\#\mathbf{E}' = 30^N$.

$$P(\mathbf{E}) = 1 - P(\mathbf{E}') = 1 - 1 - \frac{30^N}{36^N} = 1 - \left(\frac{30}{36}\right)^N$$

➤ **b.**
#**A** = 30.

For the first toss, there are 30 possibilities.

For the second toss, there are 29 remaining possibilities.

For the third toss, there are 28 remaining possibiities.

For the N toss, there are 30 - N + 1 remaining possibiities.

Therefore, the probability that all pairs of numbers are different is $\frac{30 - N + 1}{36^N}$.

16.
For 10 shots, let X equal the maximum number of misses between two hits and Y the number of hits.

Assume Y = 2.

Let **E** be the event that in 10 shots, she hits the target only twice. The following number of possibilities can happen:

Zero misses between hits. This can happen in 9 different ways.

One miss between hits. This can happen in 8 different ways.

Two misses between hits. This can happen in 7 different ways.

Three misses between hits. This can happen in 6 different ways.

Four misses between hits. This can happen in 5 different ways.

Five misses between hits. This can happen in 4 different ways.

Six misses between hits. This can happen in 3 different ways.

Seven misses between hits. This can happen in 2 different ways.

Eight misses between hits. This can happen in 1 different ways.

#**E** = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45.

17.
$\mathbf{E} = (\mathbf{K}_1 \cup \mathbf{D}_1) \cap \mathbf{K}_2 = (\mathbf{K}_1 \cap \mathbf{K}_2) \cup (\mathbf{D}_1 \cap \mathbf{K}_2)$

$\#\mathbf{E} = \#(\mathbf{K}_1 \cap \mathbf{K}_2) + \#(\mathbf{D}_1 \cap \mathbf{K}_2) - \#(\mathbf{K}_1 \cap \mathbf{D}_1 \cap \mathbf{K}_2)$

$\mathbf{D}_1 = (\mathbf{K}_1 \cap \mathbf{D}_1) \cup (\mathbf{K}_1' \cap \mathbf{D}_1)$

$\mathbf{D}_1 \cap \mathbf{K}_2 = [(\mathbf{K}_1 \cap \mathbf{D}_1) \cup (\mathbf{K}_1' \cap \mathbf{D}_1)] \cap \mathbf{K}_2 = [(\mathbf{K}_1 \cap \mathbf{D}_1) \cap \mathbf{K}_2] \cup [(\mathbf{K}_1' \cap \mathbf{D}_1) \cap \mathbf{K}_2]$

$\#(\mathbf{D}_1 \cap \mathbf{K}_2) = \#[(\mathbf{K}_1 \cap \mathbf{D}_1) \cap \mathbf{K}_2] + \#[(\mathbf{K}_1' \cap \mathbf{D}_1) \cap \mathbf{K}_2]$

$\#\mathbf{E} = \#(\mathbf{K}_1 \cap \mathbf{K}_2) + \#(\mathbf{D}_1 \cap \mathbf{K}_2) - \#(\mathbf{K}_1 \cap \mathbf{D}_1 \cap \mathbf{K}_2) = \#(\mathbf{K}_1 \cap \mathbf{K}_2) + \#[(\mathbf{K}_1 \cap \mathbf{D}_1) \cap \mathbf{K}_2] + \#[(\mathbf{K}_1' \cap \mathbf{D}_1) \cap \mathbf{K}_2]$ -

$\#(\mathbf{K}_1 \cap \mathbf{D}_1 \cap \mathbf{K}_2) = (4)(3) + 1(3) + (12)(4) - 1(3) = 60$

18.
S = {(h,h), (t,h,h),(h,t,h),(t,t,h,h),(t,h,t,h), (h,t,t.h),
(t,t,t,h,h),(t,t,h,t,h),(t,h,t,t,h),(h,t,t,t,h),(t,t,t,t,t),(h,t,t,t,t),(t,h,t,t,t),(t,t,h,t,t),(t,t,t,h,t),(t,t,t,t,h)}

19.
➤ **a.**
There are n tosses of the coin. This is sampling with replacement. Therefore,

$\#\mathbf{S} = 2^n$

Since each single element of **S** has equal chance of occurring, the probability is $1/2^n$.

➤ **b.**
E: The event that each toss alternates a different face.

E = {(h,t,h,t,...,h,t), (t,h,t,h,...,t,h)}

$\#\mathbf{E} = 2$.

Therefore, $P(\mathbf{E}) = 2/2^n = 1/2^{n-1}$

20.
➤ **a.**
There are 3 tosses of the die. This is sampling with replacement. Therefore,

$\#\mathbf{S} = 6^3 = 216$. .

Since each single element of **S** has equal chance of occurring, the probability is $1/6^3 = 1/216$.

➤ **b.**

E = {(6,5,4),(6,5,3),(6,5,2),(6,5,1),(6,4,3),(6,4,2),(6,4,1),(6,3,2),(6,3,1),(6,2,1),
(5,4,3),(5,4,2),(5,4,1),(5,3,2),(5,3,1),(5,2,1),(4,3,2),(4,3,1),(4,2,1),(3,2,1)}

#**E** = 20.

Therefore, P(**E**) = 20/216

21.

➤ **a.**

All 3 cards drawn from the deck have equal chance of being drawn.

There #**S** = (52)(51)(50) = 132600 and therefore, the probability of each single elemnt of **S** to occur has probability 1/132600.

➤ **b.**

E: The event that only 2 kings are drawn.

$\mathbf{E}_1$: The event that 2 kings are drawn on the first card drawn and the second card drawn.

$\mathbf{E}_2$: The event that 2 kings are drawn on the first card drawn and the third card drawn.

$\mathbf{E}_3$: The event that 2 kings are drawn on the second card drawn and the third card drawn.

$\mathbf{E} = \mathbf{E}_1 \cup \mathbf{E}_2 \cup \mathbf{E}_3$

$\#\mathbf{E}_1 = (4)(3)(48) = 576$

$\#\mathbf{E}_2 = (4)(48)(3) = 576$

$\#\mathbf{E}_3 = (48)(4)(3) = 576$

$\#\mathbf{E} = \#\mathbf{E}_1 + \#\mathbf{E}_2 + \#\mathbf{E}_3 = 3(4)(3)(48) = 1728$

P(**E**) = 1728/132600

22.

#S = (2)(2)(2)(2)(2)(2) = 64.

E = {(h,h,t,t,t,t),(t,h,h,t,t,t),(t,t,h,h,t,t),(t,t,t,h,h,t),(t,t,t,t,h,h)}

#**E** = 5

Since each single element has equal probability of occurring , P(**E**) = 5/64.

23.

➤ **a.**

We will use the formula sampling without replacement.

N = 10
r = 10

#**S** = (10)(9)(8)(7)(6)(5)(4)(3)(2)(1) = 3,628,800

➤ **b.**

E: The event that horses 5,8,2 will come into the money.

These three horses can come in the money (3)(2)(1) = 6.

The remaining horses will come in (7)(6)(5)(4)(3)(2)(1)

Therefore, #**E** = 6(7)(6)(5)(4)(3)(2)(1)

Therefore, P(**E**) = #**E**/#**S** = 6/(10)(9)(8) = 6/720.
